On-Shell Recursion Relations for Multi-Parton One-Loop Amplitudes

Carola F. Berger

*Stanford Linear Accelerator Center*

with

Zvi Bern, Lance Dixon, Darren Forde, David Kosower

SUSY06 – June 13th, 2006
[1] Z. Bern, L. J. Dixon, D. A. Kosower,


    to appear.

[4] CFB, V. Del Duca, L. J. Dixon,
    to appear.
SM background from $Z(\rightarrow \nu\bar{\nu}) + \text{jets}$

**Red dots:** ALPGEN, shaded histogram: PYTHIA, black dots: SUSY signal

ALPGEN based on exact LO multi-parton amplitudes, PYTHIA uses shower-algorithm $\Rightarrow$ discrepancy for events with many hard jets

$pp \rightarrow Z + 4$ jets needed at NLO!
### The (In)Famous Experimenters’ Wishlists

**Run II Monte Carlo Workshop 2001**

<table>
<thead>
<tr>
<th>Single boson</th>
<th>Diboson</th>
<th>Triboson</th>
<th>Heavy flavor</th>
</tr>
</thead>
<tbody>
<tr>
<td>$W + \leq 5j$</td>
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<td>$WWW + \leq 3j$</td>
<td>$t\bar{t} + \leq 3j$</td>
</tr>
<tr>
<td>$W + b\bar{b} + \leq 3j$</td>
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<td>$t\bar{t} + \gamma + \leq 2j$</td>
</tr>
<tr>
<td>$W + c\bar{c} + \leq 3j$</td>
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<td>$WZZ + \leq 3j$</td>
<td>$t\bar{t} + H + \leq 2j$</td>
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<tr>
<td>7. $pp \rightarrow V + 3\text{ jets}$</td>
<td>VBF $\rightarrow H \rightarrow VV$</td>
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Les Houches 2005
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Large number of high-multiplicity processes that need to be computed!
The LHC turns on in 2007!
Feynman Graphs

- Feynman rules are too general, not optimized, do not take into account all symmetries of the theory
- Vertices and propagators involve gauge-dependent off-shell states
- Explosive growth of number of diagrams/terms

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<th>one loop</th>
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<td>3,017,490</td>
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$+ 219$ more

$p^2 \neq 0$
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Time to panic??
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**Time to panic??**  —  **No!**

- **(Semi)Numerical approaches and automatization**
  - MadEvent, ALPGEN, CompHEP, GRACE, HELAC/PHEGAS, . . .
  - Kramer, Soper, Nagy; Ellis, Giele, Glover, Zanderighi; Binoth, Ciccolini, Guillet, Heinrich, Kauer, Pilon, Schubert; Czakon; Anastasiou, Daleo; . . .

- **Recursion relations**
- Strip color information, only calculate diagrams with cyclic color ordering
  \[ \Rightarrow 36 \text{ diagrams (not 220) for 6 gluons at tree level} \]
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- Decompose one-loop QCD amplitudes

\[
A_{\text{QCD}} = A_{\mathcal{N}=4} - 4 A_{\mathcal{N}=1} + A_{\text{scalar}}
\]
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  \[ A^{\text{QCD}} = A^{\mathcal{N}=4} - 4 A^{\mathcal{N}=1} + A^{\text{scalar}} \]
- Use the “right variables” to expose more symmetries - spinor helicity formalism
  Transformation to Penrose’s twistor space = “half Fourier transform” of spinors (only left-handed spinors transformed)
  \[ \Rightarrow \text{amazingly simple structure of scattering amplitudes} \]

Witten; Nair; Roiban, Spradlin, Volovich
Color Ordering, Spinors and Twistors

- Strip color information, only calculate diagrams with cyclic color ordering
  ⇒ 36 diagrams (not 220) for 6 gluons at tree level

- Decompose one-loop QCD amplitudes
  \[
  A^{QCD} = A^{\mathcal{N}=4} - 4 A^{\mathcal{N}=1} + A^{\text{scalar}}
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  Witten; Nair; Roiban, Spradlin, Volovich

- “Recycle” known amplitudes via recursion relations
  Berends, Giele; Mahlon; Cachazo, Svrcek, Witten; Britto, Cachazo, Feng, Witten
Complex continue spinors and momenta $\Rightarrow$
Amplitude function of complex parameter

$$A(z) = A(p_1, \ldots, p_j(z), p_{j+1}, \ldots, p_l(z), \ldots, p_n)$$

If $A(z \rightarrow \infty) \rightarrow 0$
Cauchy’s theorem

$$\frac{1}{2\pi i} \oint_C \frac{dz}{z} A(z) = 0$$

$A(0)$ is the physical amplitude
Recursion Relations at Tree Level

\[ A(0) = - \sum \text{poles } \alpha \ \text{Res} \ \frac{A(z)}{z} \]

\[ = \sum \text{configs} \ A_L \frac{1}{P^2} A_R \]

Poles in \( z \) correspond to physical factorizations

\[ \frac{1}{\hat{P}^2_{l...m}} = \frac{1}{P^2_{l...m} - z \langle j^- | P_{l...m} | k^- \rangle} \]

\[ A_n = \sum_k A_{n-k} \cdot A_k \]

Britto, Cachazo, Feng, Witten
On-Shell Recursions

Proof at tree level only relies on Cauchy’s theorem and basic factorization properties.

⇒ Many applications at tree level

- SUSY - processes with massless fermions
  Luo, Wen
- QCD - QCD is supersymmetric at tree level
- Massive scalars and fermions
  Badger, Glover, Khoze, Svrcek; Forde, Kosower; Schwinn, Weinzierl; Ferrario, Rodrigo, Talavera
- Higgs (top loop integrated out)
  Badger, Dixon, Glover, Khoze
- Gravity
  Bedford, Brandhuber, Spence, Travaglini; Cachazo, Svrcek; Bjerrum-Bohr, Dunbar, Ita, Perkins, Risager
QCD at One Loop - A Disaster?

Branch cuts
- Branch cuts
- Double poles, ‘unreal poles’ and nonstandard factorizations
- Branch cuts
- Double poles, ‘unreal poles’ and nonstandard factorizations
- $A(z \to \infty) \neq 0$
### Bootstrap model

From Wikipedia, the free encyclopedia

In physics, the term **bootstrap model** is used for the class of theories that assume that very general consistency criteria are sufficient to determine the whole theory completely. In such theories, typically examples of quantum field theory, it is impossible to divide the objects and concepts to elementary and composite ones. See Geoffrey Chew. This strategy turned out to be successful only in the case of two-dimensional conformal field theory where many insights can indeed be derived by this method.

## Here: very general consistency criteria

- **Cuts (unitarity)**
- **Poles (factorization)**

\[
A(z) = C(z) + R(z)
\]

(4)
Cut Parts

$Z$

$C$ ??

Cut parts contain only $L_i$, $\ln$, $\text{Li}$, cut-constructible! via (generalized) unitarity

$Z_{\text{dLIPS}}(L_1; L_2) A_{\text{tree}}(L_1; m_1; \ldots; m_2; L_2)$

Trees “recycled” into loops

Bern, Dixon, Dunbar, Kosower; Bedford, Brandhuber, McNamara, Spence, Travaglini; Quigley, Rozali; Britto, Buchbinder, Cachazo, Feng, Mastrolia; Bern, Bidder, Bjerrum-Bohr, Dixon, Dunbar, Ita, Perkins
Cut Parts

$C(0)$ contains only $\text{Li}, \ln, \pi^2$ – cut-constructible! via (generalized) unitarity

$$\int d\text{LIPS}(-l_1, l_2) A^{\text{tree}}(-l_1, m_1, .., m_2, l_2) A^{\text{tree}}(-l_2, m_2+1, .., m_1-1, l_1)$$

Trees “recycled” into loops

Bern, Dixon, Dunbar, Kosower; Bedford, Brandhuber, McNamara, Spence, Travaglini;
Quigley, Rozali; Britto, Buchbinder, Cachazo, Feng, Mastrolia; Bern, Bidder, Bjerrum-Bohr,
Dixon, Dunbar, Ita, Perkins
On-Shell Recursion for Rational Parts

\[ A(z) = C(z) + R(z) \]
\[ A(0) = C(0) + \text{Inf } R - \sum_{\text{poles } \alpha} \text{Res}_{z=z_\alpha} \frac{R(z)}{z} \]
\[ = C(0) + \text{Inf } R + \sum_{\text{configs}} A_L \frac{1}{P_l...m} A_R \]

Loops “recycled” into loops (ignoring slight subtleties with spurious singularities)

Bern, Dixon, Kosower
Non-Standard Factorizations

\[ A(0) = C(0) \]

\[ - \sum_{\text{poles } \alpha} \text{Res}_{z=z_\alpha} \frac{R(z)}{z} \]

Factorization properties unclear at one loop.
Non-Standard Factorizations

\[ A(0) = C(0) \]

\[ - \sum_{\text{poles } \alpha} \text{Res}_{z=z_\alpha} \frac{R(z)}{z} \]

\[ = \sum \left\{ \begin{array}{c}
\text{configs} \\
\end{array} \right\} \]
Non-Standard Factorizations

\[ A(0) = C(0) + \text{Inf } R - \sum_{\text{poles } \alpha} \text{Res } \frac{R(z)}{z} \]

\[ = \sum \left\{ \text{configs} \right\} \]

\[ = 0 \quad \sim \text{double pole} + \text{ratl } \times (- - +) \text{tree} \]

\[ \sim \text{rational } \times (- - +) \text{tree} \]

Factorization properties unclear at one loop.
Large-z Contributions

Can pick continuations to avoid either non-standard factorizations or \( z \to \infty \) contributions, but in general not both!

- Continuation \([j, l]\) avoids non-standard factorizations

\[
A(0) = C(0) + \text{Inf} R + R^{[j, l]}_{\text{recur}}
\]
Large-z Contributions

Can pick continuations to avoid either non-standard factorizations or $z \to \infty$ contributions, but in general not both!

- **Continuation** $[j, l]$ avoids non-standard factorizations

\begin{equation}
A(0) = C(0) + \text{Inf } R + R_{\text{recurs}}^{[j, l]}
\end{equation}

- **Continuation** $[a, b]$ has no large-parameter contributions

\begin{equation}
A(0) = C(0) + R_{\text{recurs}}^{[a, b]} + \text{non-standard channels}^{[a, b]}
\end{equation}
The Bootstrap Formalism

Solution ⇒ use two continuations!
Extract large-parameter contributions of primary continuation from auxiliary relation (6)

\[ A(0) = C(0) + R_{\text{recurs}}^{[a,b]} + \text{non-standard}^{[a,b]} + \text{Inf}_{[j,l]} \]

\[ \text{Inf } R = \text{Inf } R_{\text{recurs}}^{[a,b]} \]

\[ \text{if } \text{Inf } \text{non-standard channels}^{[a,b]}_{[j,l]} = 0 \]
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\[ A(0) = C(0) + R[a,b]_{\text{recurs}} + \text{non-standard}[a,b] \quad \text{Inf} [j,l] \]

\[ \text{Inf} R = \text{Inf} R[a,b]_{\text{recurs}}[j,l] \]

(8) \text{if} \quad \text{Inf} [\text{non-standard channels}[a,b]]_{[j,l]} = 0

The complete bootstrap

\[ A(0) = C(0) + R[j,l]_{\text{recurs}} + \text{Inf} R[a,b]_{\text{recurs}}[j,l] \]

Passes all nontrivial checks!

CFB, Bern, Dixon, Forde, Kosower
Results

- All-multiplicity formulae for \((++\ldots+)\), \((-+\ldots+)\) one-loop gluon amplitudes (also with a fermion pair)
  - Bern, Dixon, Kosower

- All-multiplicity formulae for \((+++\ldots++-\ldots+)\) one-loop gluon amplitudes
  - Forde, Kosower; CFB, Bern, Dixon, Forde, Kosower

- All-multiplicity formulae for \((-+-\ldots+)\) one-loop gluon amplitudes
  - CFB, Bern, Dixon, Forde, Kosower

- Some all-multiplicity results for parts of Higgs plus gluons (and fermion pair) at NNLO (effective theory - top loop integrated out)
  - CFB, Del Duca, Dixon

All of the above \(\ll \infty\) pages

- Working algorithm for all other configurations of one-loop gluon amplitudes!
  - CFB, Bern, Dixon, Forde, Kosower
To-Do List

- Understand complex factorization at one loop and beyond + connection to Lagrangian?
- Higher loops?
- Massive partons (external fermions, scalars, . . .)
- Automatization
- Attack the wishlists...
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