Classification of 1D and 2D Orbifolds

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Outline

- **applications**
  compactification of QFTs and string theories

- **classification and geometry**
  1D orbifolds: S^1, $S^1/\mathbb{Z}_2$
  2D orbifolds: T^2, $T^2/\mathbb{Z}_3$, $T^2/\mathbb{Z}_2''$, $\mathbb{R}P^2$

- **basis functions**
  proof of orthonormality and completeness

D. Schattschneider, Amer. Math. Monthly, Vol. 85, No. 6 (1978) 439;
A. Mück, L. N., A. Pilaftsis and R. Rückl, Phys. Rev. D 71 (2005) 066004;
D. M. Ghilencea, D. Hoover, C. P. Burgess and F. Quevedo, JHEP 0509 (2005) 050.

Beyond Standard Model Research

- hierarchy problem

  Arkani-Hamed, Dimopoulos and Dvali [PLB 429 (1998) 263]

  Randall and Sundrum [PRL 83 (1999) 3370]

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- hierarchy problem

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- neutrino physics

  seesaw mechanism $\longrightarrow$ sterile bulk ν coupling to SM brane fermions

- dark matter

  lightest supersymmetric particle (LSP) $\longrightarrow$ lightest Kaluza-Klein particle (LKP)

- dark energy

  Cosmological constant or Quintessence $\longrightarrow$ Casimir energy

- electroweak physics

  High Energy Unitarity

  Higgs exchange $\longrightarrow$ exchange of infinite tower of Kaluza-Klein modes

  A. Mück, L. N., A. Pilaftsis and R. Rückl, Phys. Rev. D 71 (2005) 066004.

Orbifolds: Definition

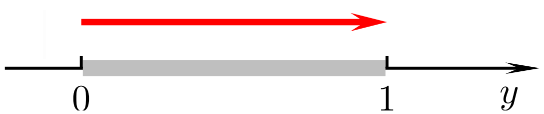
An Orbifold O is a quotient space of a manifold M modulo a finte group action Γ.

$$O = M/\Gamma$$

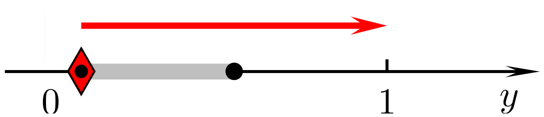
Here, we consider flat orbifolds in one and two dimensions.
The Γ are cocompact discrete groups of isometries of $\mathbb{R}$ or $\mathbb{R}^2$.

$$\mathbb{R}/\Gamma \qquad \mathbb{R}^2/\Gamma$$

1D space group Γ	**1D orbifold** $\mathbb{R}/\Gamma = S^1/\Gamma'$		
$\mathbb{Z}$	$\mathbb{R}/\mathbb{Z}$	S^1	circle
$\mathbb{D}_\infty$	$\mathbb{R}/\mathbb{D}_\infty$	$S^1/\mathbb{Z}_2$	interval



$$y \sim y+1$$



$$\begin{aligned} y &\sim y+1 \\ &\sim -y \end{aligned}$$

$$\mathbb{Z} = \langle t \rangle$$

$$\mathbb{Z}_2 = \langle r | r^2 = \mathbf{1} \rangle$$

$$\mathbb{D}_\infty = \langle t, r | r^2 = \mathbf{1}, (tr)^2 = \mathbf{1} \rangle$$

Interval $S^1/\mathbb{Z}_2 = \mathbb{R}/\mathbb{D}_\infty$

$$\mathbb{D}_\infty = \langle t, r | r^2 = \mathbf{1}, (tr)^2 = \mathbf{1} \rangle$$

$$t: \quad y \sim y+1$$

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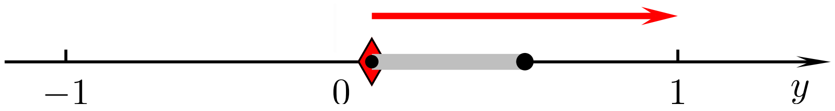

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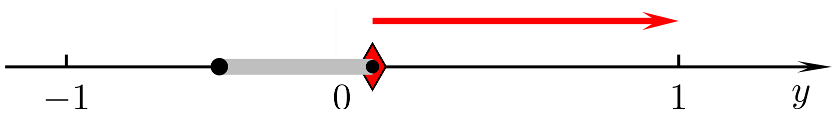

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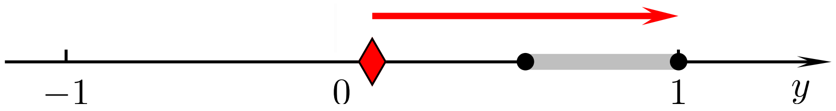

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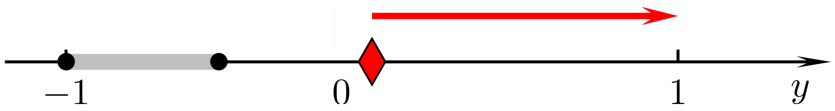

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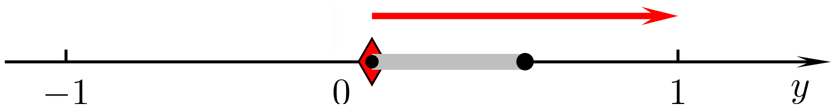

$$y \sim -y \sim 1 - y$$

$$\sim y - 1 \sim y$$

$$\Rightarrow \quad (tr)^2 = \mathbf{1}$$

Interval $S^1/\mathbb{Z}_2 = \mathbb{R}/\mathbb{D}_\infty$

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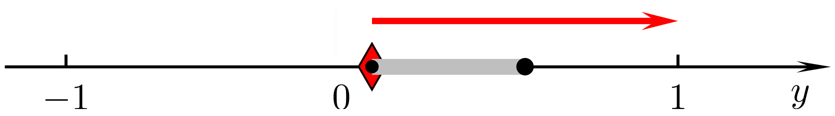


$$y \sim -y \sim 1 - y$$
$$\sim y - 1 \sim y$$
$$\Rightarrow \quad (tr)^2 = \mathbf{1}$$

$$\mathbb{D}_\infty \simeq \mathbb{Z}_2 * \mathbb{Z}_2$$
$$= \langle r | r^2 = \mathbf{1} \rangle * \langle r' | r'^2 = \mathbf{1} \rangle$$

$$r : \quad y \sim -y$$
$$r' \equiv tr : \quad y \sim 1 - y$$

The choice of generators is not unique.

$$S^1/\mathbb{Z}_2 = \mathbb{R}/(\mathbb{Z}_2 * \mathbb{Z}_2) \quad \simeq S^1/(\mathbb{Z}_2 \times \mathbb{Z}_2)$$

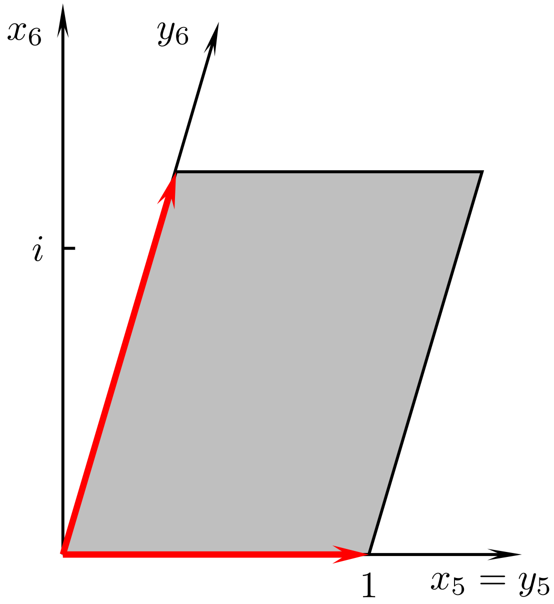
2D space group Γ	**2D orbifold** $\mathbb{R}^2/\Gamma = T^2/\Gamma'$		
p1	$\mathbb{R}^2/\mathbb{Z}^2$	T^2	torus
pg	$\mathbb{R}^2/$pg		Klein bottle
pgg	$\mathbb{R}^2/$pgg	$\mathbb{R}P^2$	real projective plane
p2	$\mathbb{R}^2/$p2	$T^2/\mathbb{Z}_2$	4-pillow
pm	$\mathbb{R}^2/(\mathbb{Z}\times\mathbb{D}_\infty)$	$T^2/\mathbb{Z}_2'$	annulus
cm	$\mathbb{R}^2/$cm	$T^2/\mathbb{Z}_2''$	Möbius strip
pmm	$\mathbb{R}^2/\mathbb{D}_\infty^2$	$T^2/\mathbb{D}_2$	rectangle
cmm	$\mathbb{R}^2/$cmm	$T^2/\mathbb{D}_2'$	triangle
pmg	$\mathbb{R}^2/$pmg	$T^2/\mathbb{F}_2$	open 4-pillow
p3	$\mathbb{R}^2/$p3	$T^2/\mathbb{Z}_3$	3-pillow
p3m1	$\mathbb{R}^2/$p3m1	$T^2/\mathbb{D}_3$	triangle
p31m	$\mathbb{R}^2/$p31m	$T^2/\mathbb{F}_3$	open 3-pillow
p4	$\mathbb{R}^2/$p4	$T^2/\mathbb{Z}_4$	3-pillow
p4m	$\mathbb{R}^2/$p4m	$T^2/\mathbb{D}_4$	triangle
p4g	$\mathbb{R}^2/$p4g		open 3-pillow
p6	$\mathbb{R}^2/$p6	$T^2/\mathbb{Z}_6$	3-pillow
p6m	$\mathbb{R}^2/$p6m	$T^2/\mathbb{D}_6$	triangle

Torus $T^2 = \mathbb{R}^2/\mathsf{p1} = \mathbb{R}^2/\mathbb{Z}^2$

$$\begin{aligned} \mathsf{p1} &\simeq \mathbb{Z}^2 \\ &= \langle t_1 \rangle \times \langle t_2 \rangle \end{aligned}$$

$$t_1 : \quad z \sim z + 1$$

$$t_2 : \quad z \sim z + \omega \qquad \omega = re^{i\theta}$$



Parameters r and θ specify shape of the torus. Compactification radii:

$$\begin{aligned} R_1 &= 1/(2\pi) \\ R_2 &= rR_1 \end{aligned}$$

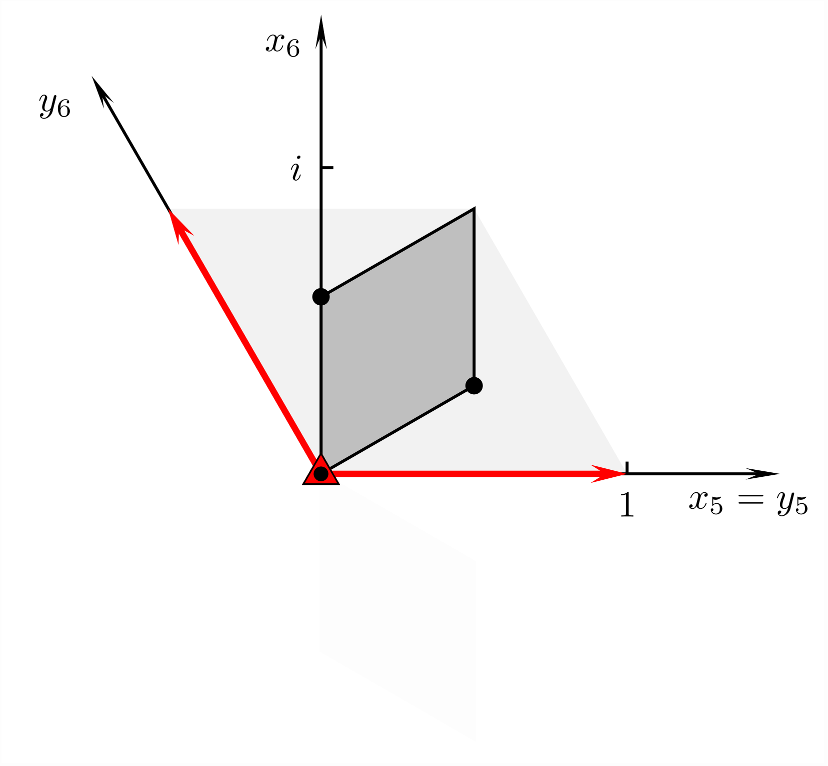
Orbifold $T^2/\mathbb{Z}_3 = \mathbb{R}^2/\mathrm{p3}$

$$\mathrm{p3} = \langle t_1, t_2, r | r^3 = (t_1 r)^3 = (t_2 r^2)^3 = \mathbf{1}, \quad r t_1 = t_2 r, [t_1, t_2] = 0 \rangle$$

$$t_1 : \quad z \sim z + 1$$

$$t_2 : \quad z \sim z + \omega$$

$$r : \quad z \sim \omega z \qquad \omega = e^{i2\pi/3}$$



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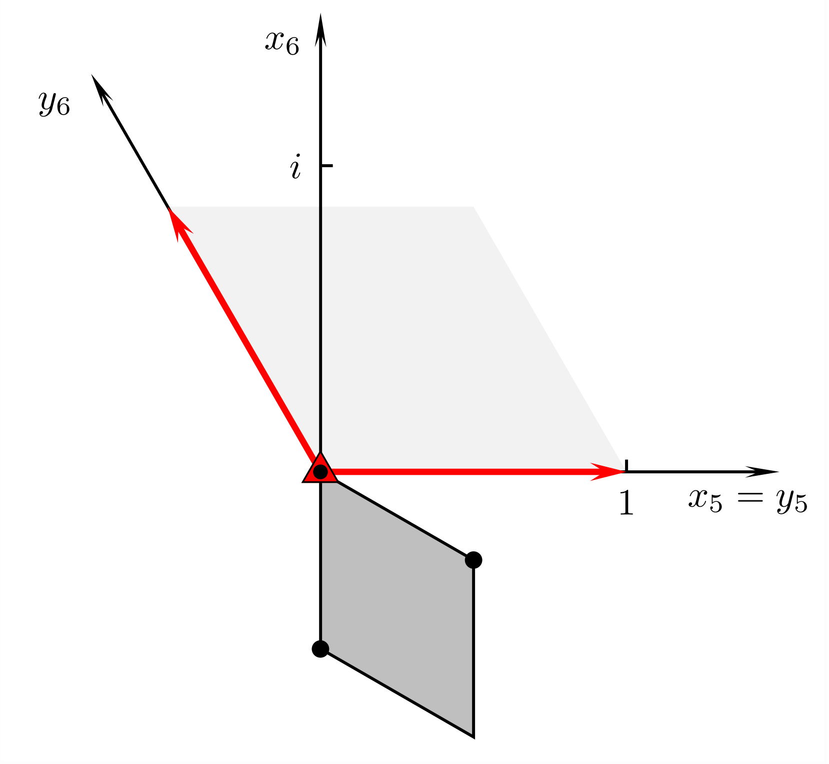

$$z \sim \omega^2 z$$

Orbifold $T^2/\mathbb{Z}_3 = \mathbb{R}^2/\text{p3}$

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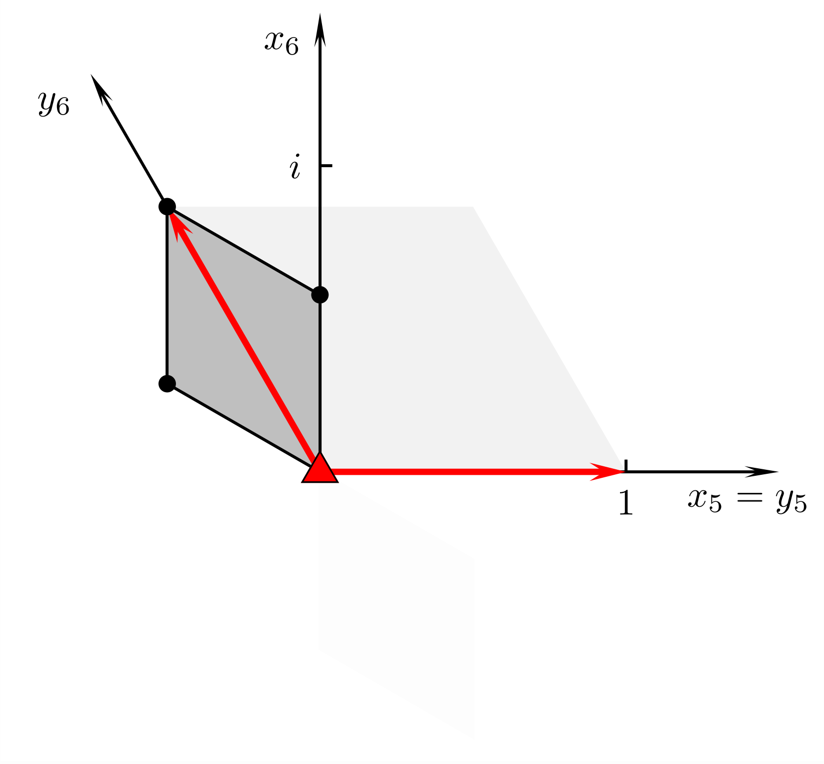

$$z \sim \omega^2 z \\ \sim \omega^2 z + \omega$$

Orbifold $T^2/\mathbb{Z}_3 = \mathbb{R}^2/\text{p3}$

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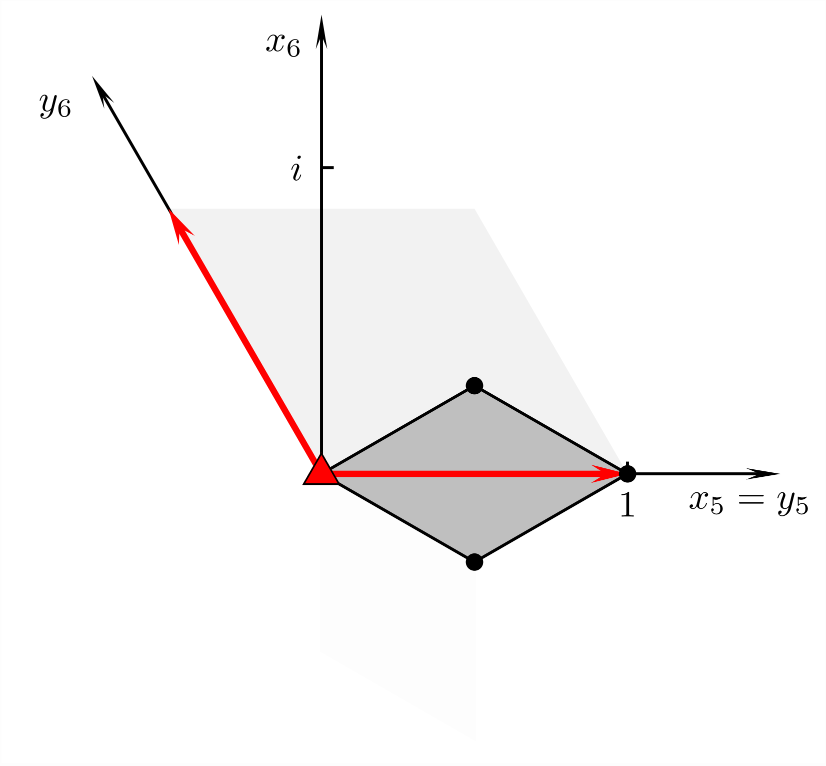

$$\begin{aligned} z &\sim \omega^2 z \\ &\sim \omega^2 z + \omega \\ &\sim \omega z + 1 \end{aligned}$$

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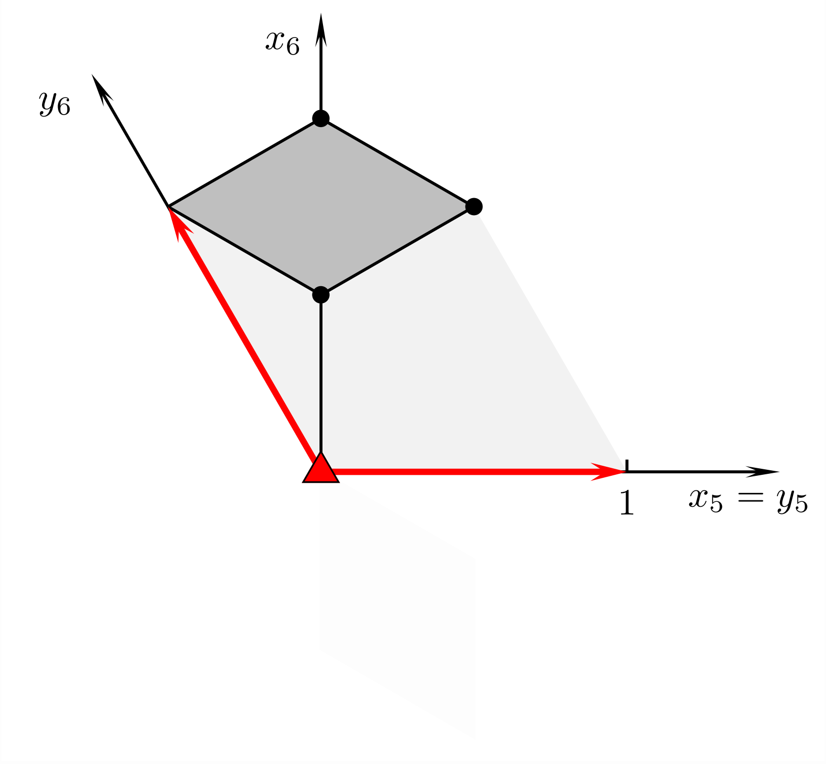

$$\begin{aligned} z &\sim \omega^2 z \\ &\sim \omega^2 z + \omega \\ &\sim \omega z + 1 \\ &\sim \omega z - \omega^2 \end{aligned}$$

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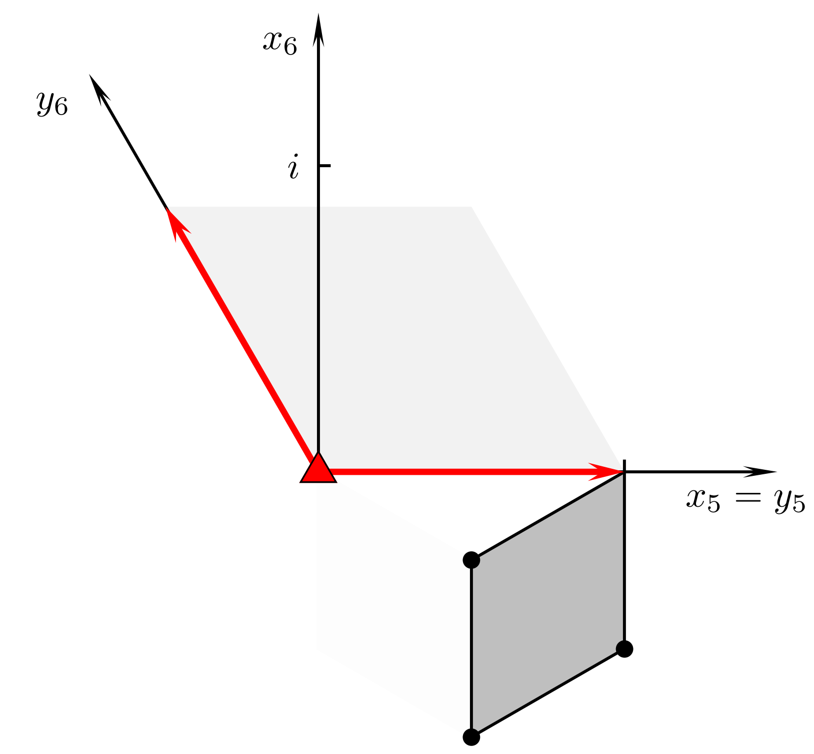

$$\begin{aligned} z &\sim \omega^2 z \\ &\sim \omega^2 z + \omega \\ &\sim \omega z + 1 \\ &\sim \omega z - \omega^2 \\ &\sim z - \omega \end{aligned}$$

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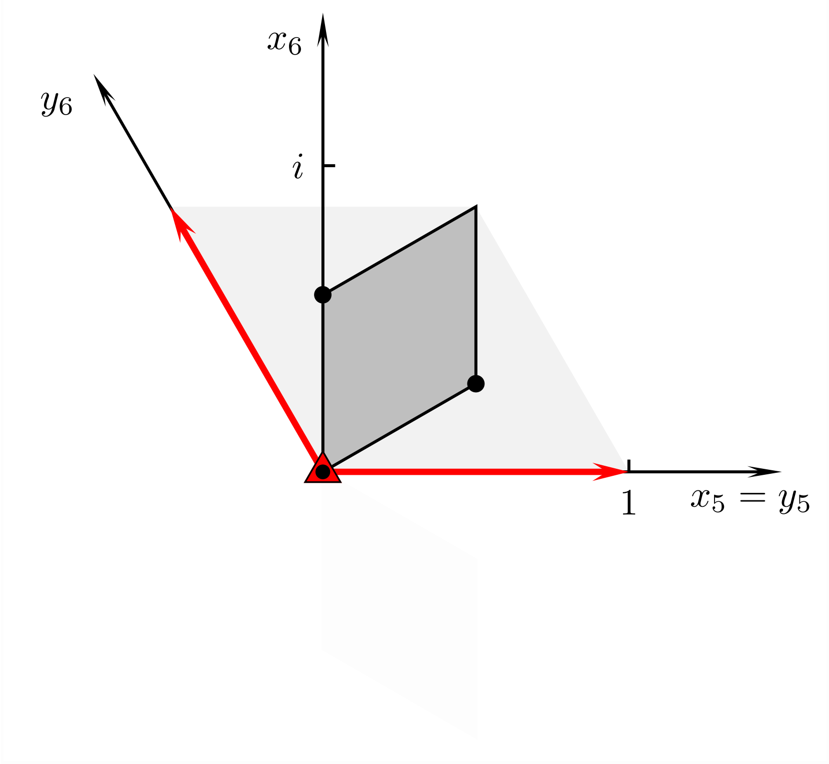

$$\begin{aligned} z &\sim \omega^2 z \\ &\sim \omega^2 z + \omega \\ &\sim \omega z + 1 \\ &\sim \omega z - \omega^2 \\ &\sim z - \omega \\ &\sim z \qquad \Rightarrow \quad (t_2 r^2)^3 = \mathbf{1} \end{aligned}$$

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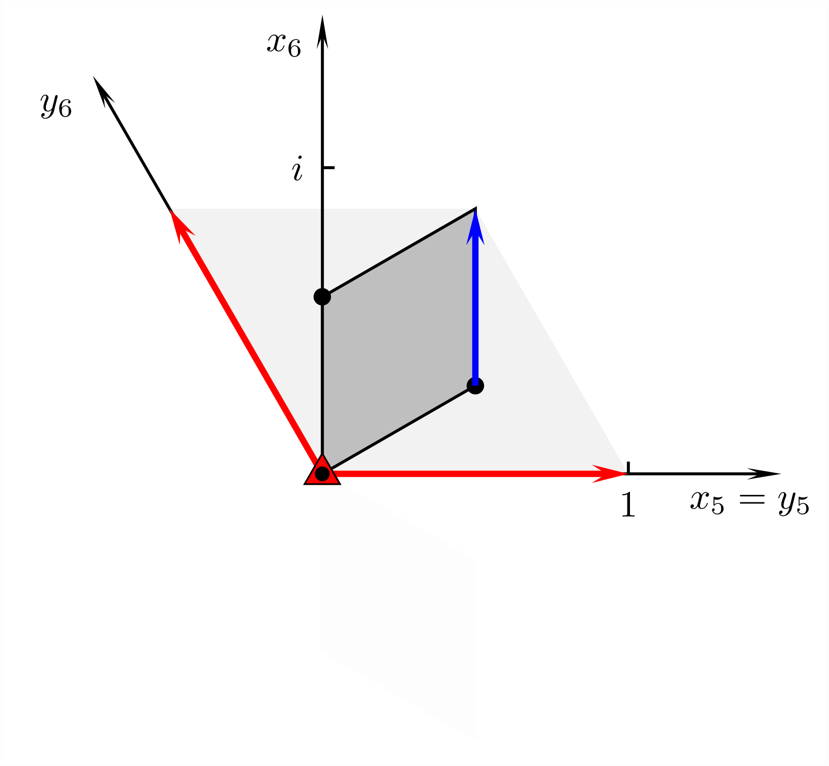

$$\begin{aligned} z &\sim \omega^2 z \\ &\sim \omega^2 z + \omega \\ &\sim \omega z + 1 \\ &\sim \omega z - \omega^2 \\ &\sim z - \omega \\ &\sim z \qquad \Rightarrow \quad (t_2 r^2)^3 = \mathbf{1} \end{aligned}$$

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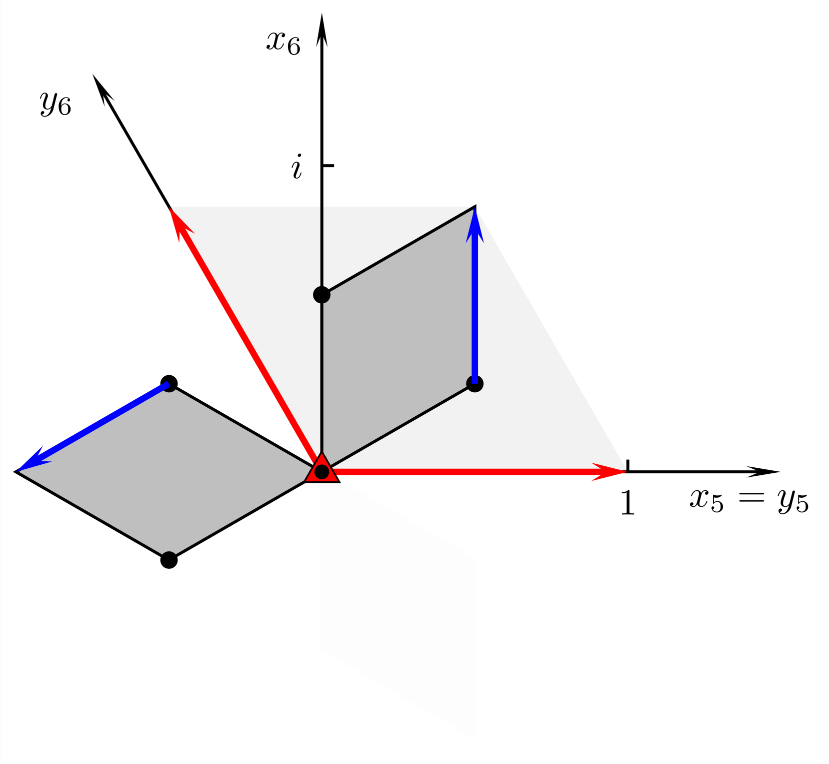

$$\begin{aligned} z &\sim \omega^2 z \\ &\sim \omega^2 z + \omega \\ &\sim \omega z + 1 \\ &\sim \omega z - \omega^2 \\ &\sim z - \omega \\ &\sim z \qquad \Rightarrow \qquad (t_2 r^2)^3 = \mathbf{1} \end{aligned}$$

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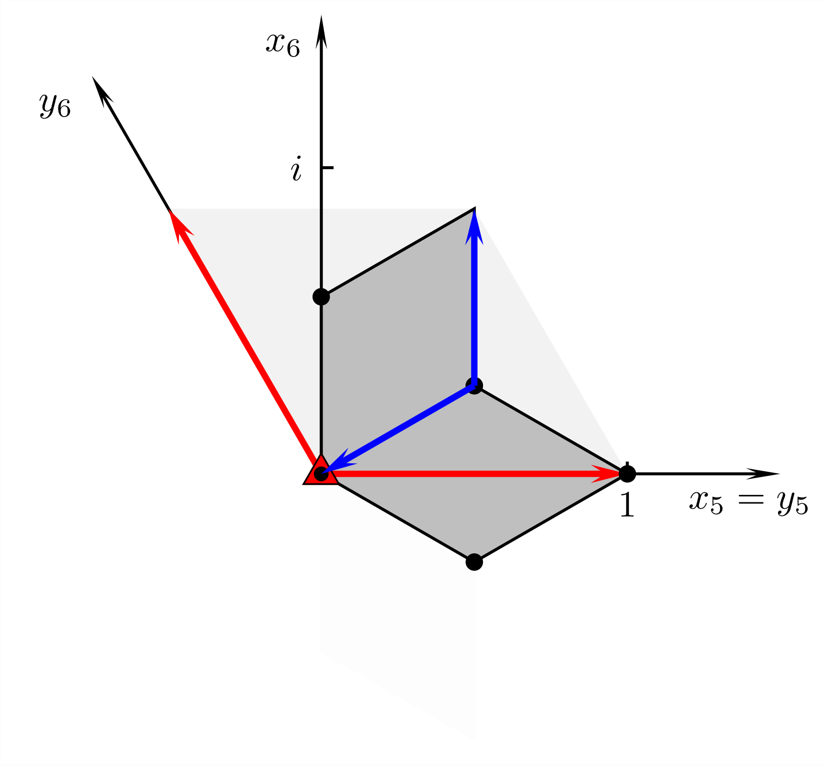

$$\begin{aligned} z &\sim \omega^2 z \\ &\sim \omega^2 z + \omega \\ &\sim \omega z + 1 \\ &\sim \omega z - \omega^2 \\ &\sim z - \omega \\ &\sim z \qquad \Rightarrow \quad (t_2 r^2)^3 = \mathbf{1} \end{aligned}$$

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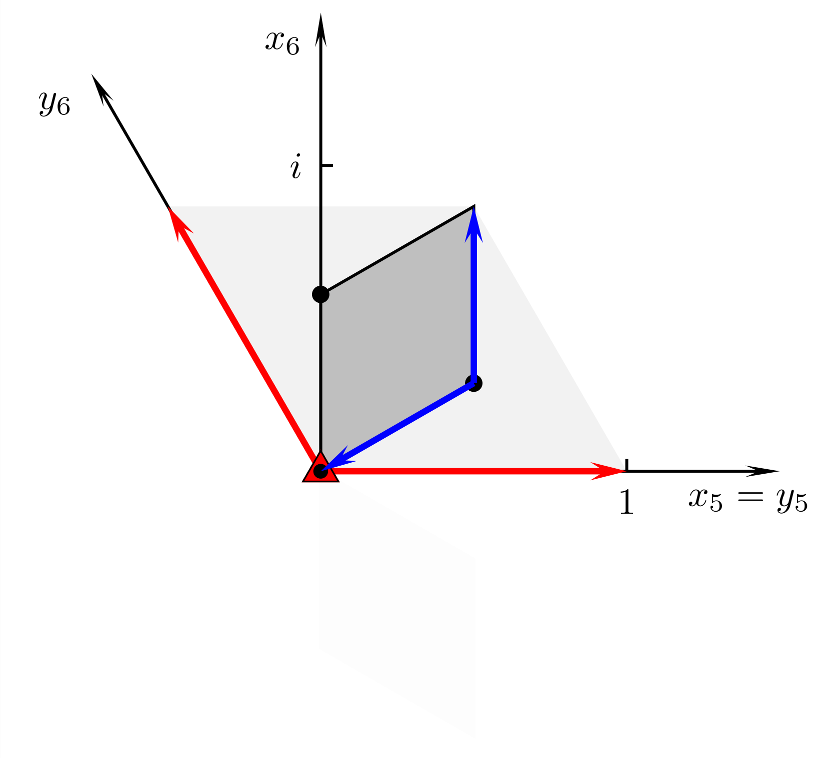
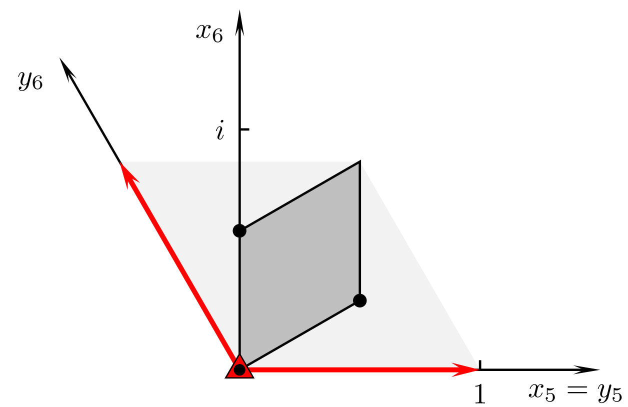


$$\begin{aligned} z &\sim \omega^2 z \\ &\sim \omega^2 z + \omega \\ &\sim \omega z + 1 \\ &\sim \omega z - \omega^2 \\ &\sim z - \omega \\ &\sim z \qquad \Rightarrow \quad (t_2 r^2)^3 = \mathbf{1} \end{aligned}$$

Alhambra mosaic
with p3 symmetry

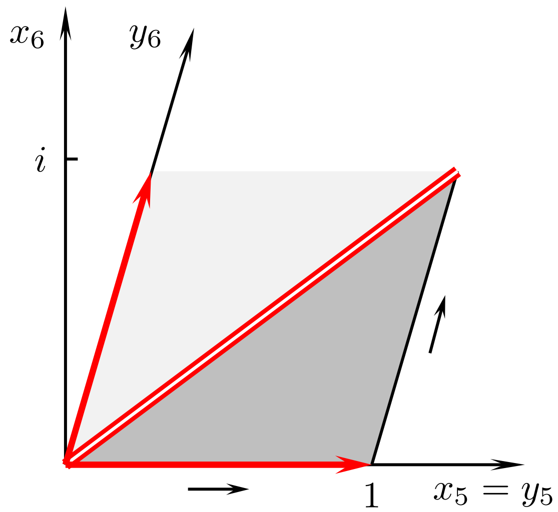


$$T^2/\mathbb{Z}_3 = \mathbb{R}^2/\text{p3}$$

Möbius strip $T^2/\mathbb{Z}_2'' = \mathbb{R}^2/\text{cm}$

$$\text{cm} = \langle t_1, t_2, f \,|\, f^2 = \mathbf{1},$$
$$f t_1 = t_2 f, [t_1, t_2] = 0\rangle$$

$$t_1 : \quad z \sim z + 1$$
$$t_2 : \quad z \sim z + \omega$$
$$f : \quad z \sim \omega z^* \qquad \omega = e^{i\theta}$$



$$\mathbb{Z}_2 \subseteq \text{cm} \quad \Rightarrow \quad T^2/\mathbb{Z}_2'' = \mathbb{R}^2/\text{cm}$$

In general:

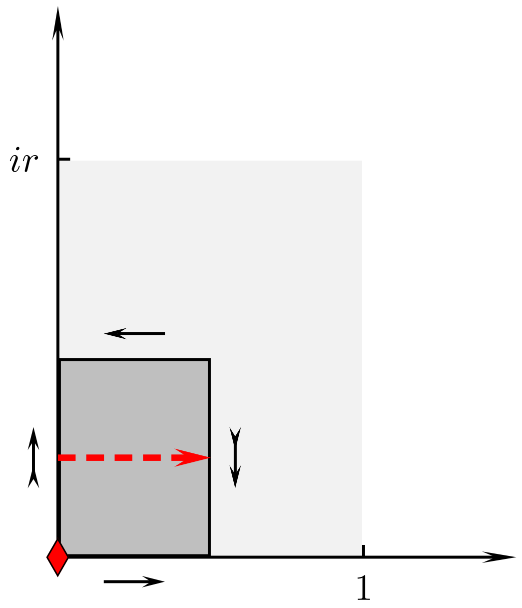
$$\Gamma' \subseteq \Gamma \quad \Rightarrow \quad T^2/\Gamma' = \mathbb{R}^2/\Gamma$$

Real projective plane $\mathbb{R}P^2 = \mathbb{R}^2/\text{pgg}$

$$\text{pgg} = \langle r, g | r^2 = (g^2 r)^2 = \mathbf{1} \rangle$$

$$r: \quad z \sim -z$$

$$g: \quad z \sim z^* + \frac{ir}{2} + \frac{1}{2}$$



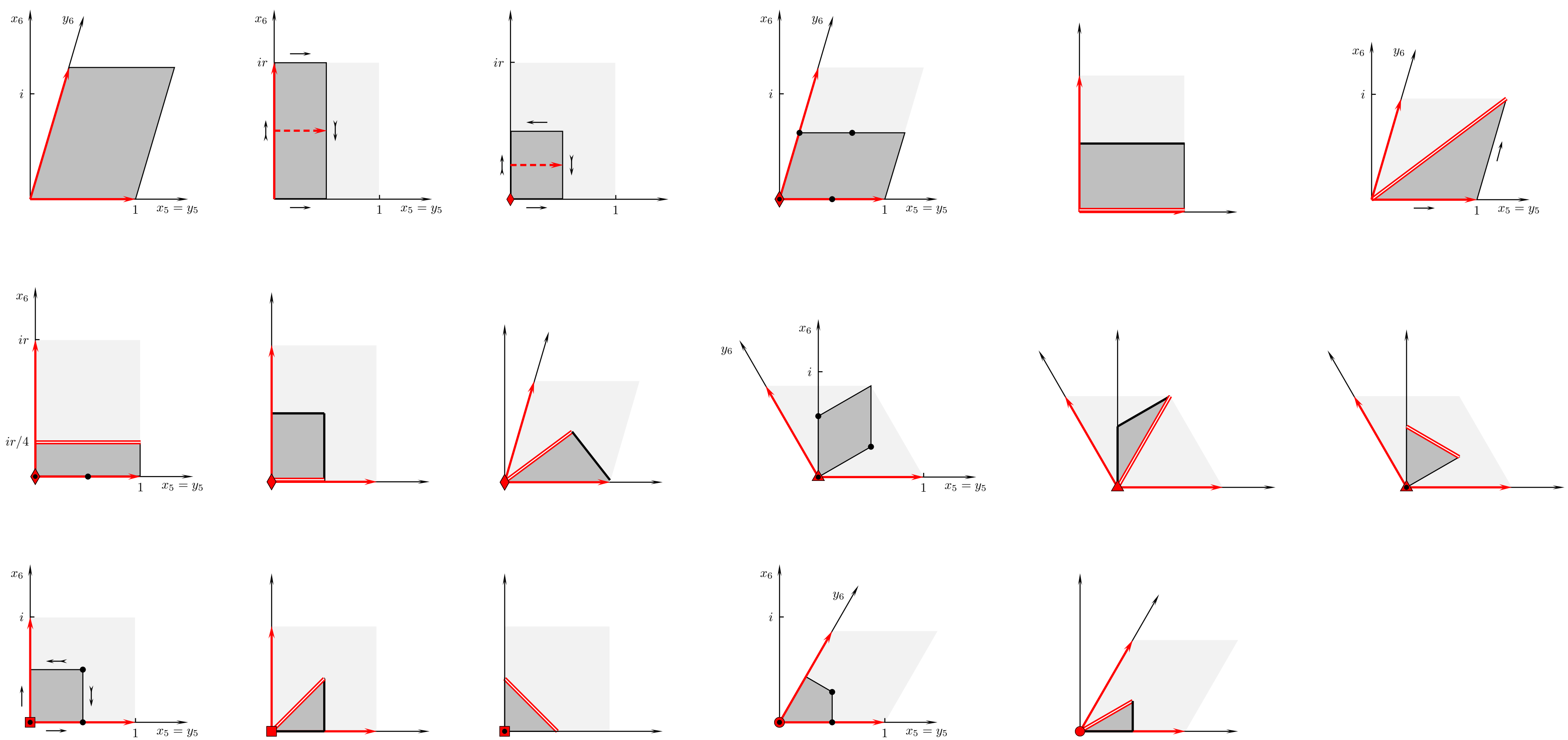
$$t_1 = g^2: \quad z \sim z + 1$$

$$t_2 = [gr]^2: \quad z \sim z + ir$$

Translations t_1 and t_2 are not generators, but are themselves generated. $\Rightarrow$

$\mathbb{R}P^2$ cannot be expressed as T^2/Γ'.

Complete list of 2D orbifolds



$$\mathsf{p1} \simeq \mathbb{Z}^2 = \langle t_1 \rangle \times \langle t_2 \rangle$$

$$\mathsf{pg} = \langle t_2, g | [g^2, t_2] = 0, t_2 g t_2 g^{-1} = \mathbf{1} \rangle$$

$$\mathsf{pgg} = \langle r, g | r^2 = (g^2 r)^2 = \mathbf{1} \rangle$$

$$\mathsf{p2} = \langle t_1, t_2, r | r^2 = (t_1 r)^2 = (r t_2)^2 = \mathbf{1}, [t_1, t_2] = 0 \rangle$$

$$\mathsf{pm} \simeq \mathbb{Z} \times \mathbb{D}_\infty = \langle t_1 \rangle \times \langle t_2, f | f^2 = (t_2 f)^2 = \mathbf{1} \rangle$$

$$\mathsf{cm} = \langle t_1, t_2, f | f^2 = \mathbf{1}, [t_1, t_2] = 0, f t_1 = t_2 f \rangle$$

$$\mathsf{pmm} \simeq \mathbb{D}_\infty^2 = \langle t_2, f | f^2 = (t_2 f)^2 = \mathbf{1} \rangle \times \langle t_1, t_2, r | r^2 = (t_1 t_2 r)^2 = \mathbf{1} \rangle$$

$$\mathsf{cmm} = \langle t_1, t_2, r, f | r^2 = f^2 = (f r)^2 = (t_1 r)^2 = (r t_2)^2 = \mathbf{1}, [t_1, t_2] = 0, f t_1 = t_2 f \rangle$$

$$\mathsf{p3} = \langle t_1, t_2, r | r^3 = (t_1 r)^3 = (t_2 r^2)^3 = \mathbf{1}, r t_1 = t_2 r, [t_1, t_2] = 0 \rangle$$

$$\mathsf{p3m1} = \langle t_1, t_2, r, f | r^3 = f^2 = (f r)^2 = (t_1 r)^3 = (t_2 r^2)^3 = \mathbf{1}, \\ r t_1 = t_2 r, f t_1 = t_2 f, [t_1, t_2] = 0 \rangle \qquad \cdots$$

So far **no structure visible** in algebraic definition of 2D space groups.

Basis functions

The basis functions are simple superpositions of exponentials.

- Algebraic definition of space groups essential for the derivation of all possible parities on the orbifold.
- Summation over all independent Kaluza-Klein modes important. Overcounting leads to incorrect results!
- Orthonormality and completeness of the basis are key to a number of proofs.

Basis functions on the circle $S^1 = \mathbb{R}/\mathbb{Z}$

Consider a complex field $\varphi(y)$ with a Scherk-Schwarz phase p.

$$\varphi(y+1) = p\,\varphi(y) \qquad \text{with} \quad p \equiv e^{i2\pi\rho},\ \rho \in [0,1) \subset \mathbb{Q}$$

$$\varphi(y) = \sum_{k=-\infty}^{\infty} \varphi_{(k)}\, f_k^{(p)}(y)$$

$$f_k^{(p)}(y) = e^{i2\pi(k+\rho)y}$$

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$$\varphi(y) = \sum_{k=-\infty}^{\infty} \varphi_{(k)}\, f_k^{(p)}(y)$$

$$f_k^{(p)}(y) = e^{i2\pi(k+\rho)y}$$

$$\int_0^j \frac{dy}{j}\, f_k^{(p)}(y) f_l^{(q)*}(y) = \delta_{k,l}\delta_{p,q} \qquad \text{with} \quad j \in \mathbb{N},\ p^j = q^j = 1$$

$$\left[\partial_5^2 + m_k^{(p)2}\right] f_k^{(p)}(y) = 0 \qquad \text{with} \quad \partial_5 \equiv \frac{\partial}{\partial y},\ \ m_k^{(p)} \equiv 2\pi(k+\rho)$$

$$f_k^{(p)}(-y) = f_{-k}^{(p^*)}(y)$$

Basis functions on the interval $S^1/\mathbb{Z}_2 = \mathbb{R}/\mathbb{D}_\infty$

$$\mathbb{D}_\infty = \langle t, r | r^2 = \mathbf{1}, (tr)^2 = \mathbf{1} \rangle \qquad \Rightarrow \qquad p^2 = 1,\ (qp)^2 = 1$$

$$\Rightarrow \qquad p,\ q = \pm$$

$$\varphi(-y) = p\ \varphi(y)$$

$$\varphi(y+1) = q\ \varphi(y)$$

Basis functions on the interval $S^1/\mathbb{Z}_2 = \mathbb{R}/\mathbb{D}_\infty$

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$$\Rightarrow \quad p,\ q = \pm$$

$$\varphi(-y) = p\, \varphi(y)$$
$$\varphi(y+1) = q\, \varphi(y)$$

The field $\varphi(y)$ can posses four parities (p, q) and be expanded in terms of $F_k^{(p,q)}(y)$.

$$F_k^{(p,q)}(y) = c_k^{(p,q)} \left[f_k^{(q)}(y) + p\, f_k^{(q)}(-y) \right]$$
$$= c_k^{(p,q)} \left[f_k^{(q)}(y) + p\, f_{-k}^{(q)}(y) \right]$$

$$q^2 = 1 \ \Rightarrow \qquad \int_0^2 \frac{dy}{2}\, F_k^{(p,q)}(y)\, F_l^{(r,s)*}(y) = \delta_{k,l}\, \delta_{p,r}\, \delta_{q,s}$$

Basis functions on the interval $S^1/\mathbb{Z}_2 = \mathbb{R}/\mathbb{D}_\infty$, continued

$$F_k^{(p,q)}(y) = \begin{cases} \sqrt{2^{1-\delta_{k,0}}}\ \cos(2k\pi y) & \text{for} \quad (p,q) = (+,+) \\ \sqrt{2^{1-\delta_{k,0}}}\ \cos([2k+1]\pi y) & \text{for} \quad (p,q) = (+,-) \\ \sqrt{2}i\ \sin(2k\pi y) & \text{for} \quad (p,q) = (-,+) \\ \sqrt{2^{1-\delta_{k,0}}}i\ \sin([2k+1]\pi y) & \text{for} \quad (p,q) = (-,-) \end{cases}$$

$$f_k^{(p)}(-y) = f_{-k}^{(p^*)}(y) \quad \Rightarrow \qquad F_k^{(\pm,q)}(-y) = F_{-k}^{(\pm,q)}(y)$$

Basis functions on the interval $S^1/\mathbb{Z}_2 = \mathbb{R}/\mathbb{D}_\infty$, continued

$$F_k^{(p,q)}(y) = \begin{cases} \sqrt{2^{1-\delta_{k,0}}}\ \cos(2k\pi y) & \text{for} \quad (p,q) = (+,+) \\ \sqrt{2^{1-\delta_{k,0}}}\ \cos([2k+1]\pi y) & \text{for} \quad (p,q) = (+,-) \\ \sqrt{2}i\ \sin(2k\pi y) & \text{for} \quad (p,q) = (-,+) \\ \sqrt{2^{1-\delta_{k,0}}}i\ \sin([2k+1]\pi y) & \text{for} \quad (p,q) = (-,-) \end{cases}$$

$$f_k^{(p)}(-y) = f_{-k}^{(p^*)}(y) \quad \Rightarrow \qquad F_k^{(\pm,q)}(-y) = F_{-k}^{(\pm,q)}(y) = \pm F_k^{(\pm,q)}(y)$$

Since $F_{-k}^{(p,q)}(y)$ and $F_k^{(p,q)}(y)$ are not independent, we restrict the indices $k \geq 0$.

$$\varphi(y) = \sum_{k=0}^{\infty} \varphi_{(k)} F_k^{(p,q)}(y)$$

Basis functions on the interval $S^1/\mathbb{Z}_2 = \mathbb{R}/\mathbb{D}_\infty$, continued

$$\int_0^2 \frac{dy}{2}\, \varphi(y)\, \delta^*(y - y') = \varphi(y')$$

$$\delta(y) = \sum_{k=0}^{\infty} d_k\, F_k^{(+,+)}(y)$$

Basis functions on the interval $S^1/\mathbb{Z}_2 = \mathbb{R}/\mathbb{D}_\infty$, continued

$$\int_0^2 \frac{dy}{2}\, \varphi(y)\, \delta^*(y-y') = \varphi(y')$$

$$\delta(y) = \sum_{k=0}^{\infty} F_k^{(+,+)*}(0)\, F_k^{(+,+)}(y)$$

$$= \sum_{k=0}^{\infty} 2^{1-\delta_{k,0}} \cos(2\pi k y)$$

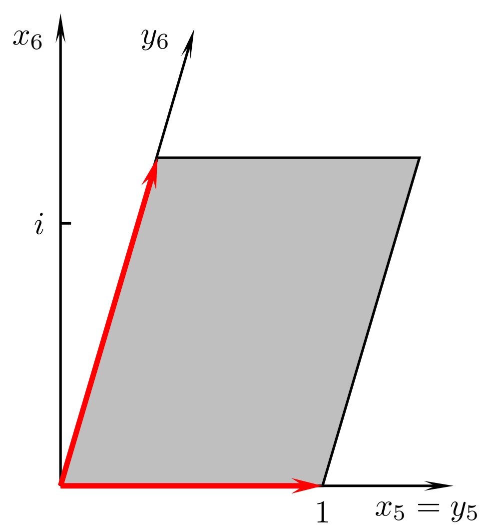
$$\delta(y_1 - y_2) = \sum_{p,q=\pm} \sum_{n=0}^{\infty} F_n^{(p,q)}(y_1)\, F_n^{(p,q)*}(y_2)$$

The basis functions $F_k^{(p,q)}(y)$ are **orthonormal** and **complete**.

Basis functions on the torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$

Here, consider a complex field $\varphi(z)$ without Scherk-Schwarz phases.

$$\varphi(z+1) = \varphi(z)$$
$$\varphi(z+\omega) = \varphi(z) \qquad \text{with} \quad \omega = r\, e^{i\theta}$$



$$z = x_5 + i x_6$$
$$= y_5 + \omega y_6$$

$$\varphi(z) = \sum_{k,l=-\infty}^{\infty} \varphi_{(k,l)}\, f_{k,l}(z)$$

$$f_{k,l}(z) = \exp\left[i2\pi(k y_5 + l y_6)\right]$$
$$= \exp\left[i2\,\mathrm{Re}(M_{k,l} z)\right] \qquad \text{with} \quad M_{k,l} = 2\pi\,\frac{l - \omega^* k}{\omega - \omega^*}$$

Basis functions on the torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$, continued

The basis functions $f_{k,l}(z)$ are **orthonormal**.

$$\int_0^1 \int_0^1 \frac{dy_5 dy_6}{r\,\sin\theta}\, f_{k,l}(y_5, y_6)\, f^*_{m,n}(y_5, y_6) = \delta_{k,l}\delta_{m,n}$$

$$\int \frac{i\,dz\,dz^*}{2(\mathrm{Im}\,\omega)^2}\, f_{k,l}(z)\, f^*_{m,n}(z) = \delta_{k,l}\delta_{m,n}$$

Basis functions on the torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$, continued

The basis functions $f_{k,l}(z)$ are **orthonormal**.

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$$\int \frac{i\,dz\,dz^*}{2(\mathrm{Im}\,\omega)^2}\, f_{k,l}(z)\, f^*_{m,n}(z) = \delta_{k,l}\delta_{m,n}$$

We find the following **mass spectrum**.

$$\begin{aligned}\left[\partial_z \partial_{z^*} + m^2_{k,l}\right] f_{k,l}(z) &= 0 \\ m^2_{k,l} &\equiv M_{k,l} M^*_{k,l} \\ &= \frac{1}{\sin^2\theta}\left[\frac{k^2}{4R_1^2} + \frac{l^2}{4R_2^2} - \frac{\cos\theta}{2R_1R_2} kl\right]\end{aligned}$$

Basis functions on $T^2/\mathbb{Z}_3 = \mathbb{R}^2/\mathrm{p3}$

$$\mathrm{p3} = \langle t_1, t_2, r | r^3 = (t_1 r)^3 = (t_2 r^2)^3 = \mathbf{1},\ r t_1 = t_2 r,\ [t_1, t_2] = 0 \rangle$$

$$\Rightarrow \quad p \in \{+, \omega, \omega^2\},\ q = s \in \{+, \omega, \omega^2\} \qquad \text{with} \quad \omega \equiv e^{i2\pi/3}$$

$$\varphi(\omega z) = p\, \varphi(z)$$

$$\varphi(z+1) = q\, \varphi(z)$$

$$\varphi(z+\omega) = s\, \varphi(z)$$

The field $\varphi(z)$ can posses nine parities (p, q). Here, we do not consider Scherk-Schwarz phases, $q = s = +$, and expand in terms of $F^{(p)}_{k,l}(z)$.

Basis functions on $T^2/\mathbb{Z}_3 = \mathbb{R}^2/\text{p3}$

$$\text{p3} = \langle t_1, t_2, r | r^3 = (t_1 r)^3 = (t_2 r^2)^3 = \mathbf{1},\ r t_1 = t_2 r,\ [t_1, t_2] = 0 \rangle$$

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The field $\varphi(z)$ can posses nine parities (p, q). Here, we do not consider Scherk-Schwarz phases, $q = s = +$, and expand in terms of $F^{(p)}_{k,l}(z)$.

$$f_{k,l}(z) = \exp\left[i2\,\text{Re}(M_{k,l} z)\right] \quad \text{with} \quad M_{k,l} = \pi\left[k - i(2l+k)/\sqrt{3}\right]$$

$$f_{k,l}(\omega z) = f_{l,-k-l}(z)$$

$$(k,l) \ \rightarrow\ (l, -k-l) \ \rightarrow\ (-k-l, k) \ \rightarrow\ (k,l)$$

Basis functions on $T^2/\mathbb{Z}_3 = \mathbb{R}^2/\mathrm{p3}$, continued

$$F_{k,l}^{(p)}(z) = \; c_{k,l}^{(p)} \, \left[f_{k,l}(z) + p^2 \, f_{k,l}(\omega z) + p \, f_{k,l}(\omega^2 z)\right]$$

Basis functions on $T^2/\mathbb{Z}_3 = \mathbb{R}^2/\text{p3}$, continued

$$F_{k,l}^{(p)}(z) = \; c_{k,l}^{(p)} \; \left[f_{k,l}(z) + p^2 \, f_{l,-k-l}(z) + p \, f_{-k-l,k}(z) \right]$$

Basis functions on $T^2/\mathbb{Z}_3 = \mathbb{R}^2/\mathrm{p3}$, continued

$$F^{(p)}_{k,l}(z) = \sqrt{3^{-1-\delta_{k,0}\delta_{l,0}}}\,\left[f_{k,l}(z) + p^2\, f_{l,-k-l}(z) + p\, f_{-k-l,k}(z)\right]$$

$$\int \frac{i\,dz\,dz^*}{2(\mathrm{Im}\,\omega)^2}\, F^{(p)}_{k,l}(z)\; F^{(q)*}_{m,n}(z) = \delta_{p,q}\,\delta_{k,m}\,\delta_{l,n}$$

Basis functions on $T^2/\mathbb{Z}_3 = \mathbb{R}^2/\text{p3}$, continued

$$F^{(p)}_{k,l}(z) = \sqrt{3^{-1-\delta_{k,0}\delta_{l,0}}}\,\left[f_{k,l}(z) + p^2\, f_{l,-k-l}(z) + p\, f_{-k-l,k}(z)\right]$$

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$$F^{(p)}_{k,l}(\omega z) = F^{(p)}_{l,-k-l}(z) = p\, F^{(p)}_{k,l}(z)$$

We restrict the indices to $k, l \geq 0$ and $k, l \leq -1$.

Basis functions on $T^2/\mathbb{Z}_3 = \mathbb{R}^2/\mathrm{p3}$, continued

$$F_{k,l}^{(p)}(z) = \sqrt{3^{-1-\delta_{k,0}\delta_{l,0}}}\,\left[f_{k,l}(z) + p^2\, f_{l,-k-l}(z) + p\, f_{-k-l,k}(z)\right]$$

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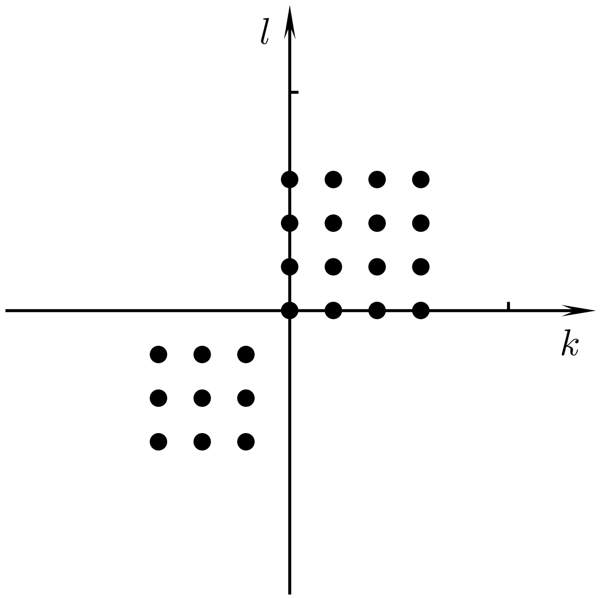

$$F_{k,l}^{(p)}(\omega z) = F_{l,-k-l}^{(p)}(z) = p\, F_{k,l}^{(p)}(z)$$

We restrict the indices to $k, l \geq 0$ and $k, l \leq -1$.

(k, l)

Basis functions on $T^2/\mathbb{Z}_3 = \mathbb{R}^2/\text{p3}$, continued

$$F^{(p)}_{k,l}(z) = \sqrt{3^{-1-\delta_{k,0}\delta_{l,0}}}\,\left[f_{k,l}(z) + p^2\, f_{l,-k-l}(z) + p\, f_{-k-l,k}(z)\right]$$

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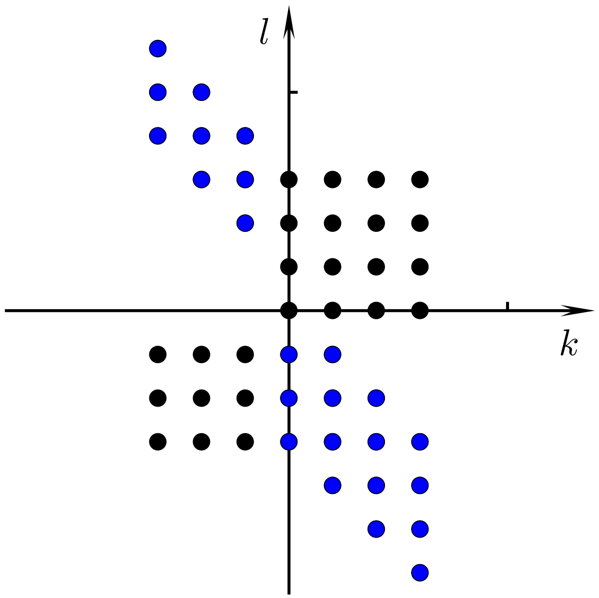

$$F^{(p)}_{k,l}(\omega z) = F^{(p)}_{l,-k-l}(z) = p\, F^{(p)}_{k,l}(z)$$

We restrict the indices to $k,l \geq 0$ and $k,l \leq -1$.

$$(k,l) \;\rightarrow\; (l,-k-l)$$

Basis functions on $T^2/\mathbb{Z}_3 = \mathbb{R}^2/\mathrm{p3}$, continued

$$F_{k,l}^{(p)}(z) = \sqrt{3^{-1-\delta_{k,0}\delta_{l,0}}}\,\left[f_{k,l}(z) + p^2\, f_{l,-k-l}(z) + p\, f_{-k-l,k}(z)\right]$$

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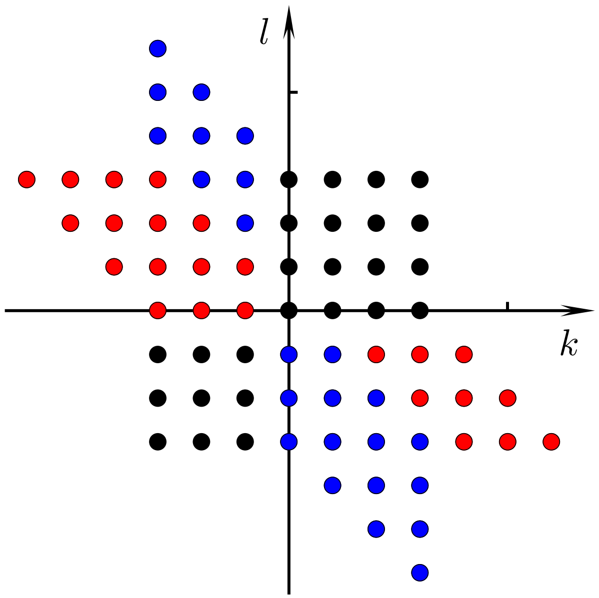


$$F_{k,l}^{(p)}(\omega z) = F_{l,-k-l}^{(p)}(z) = p\, F_{k,l}^{(p)}(z)$$

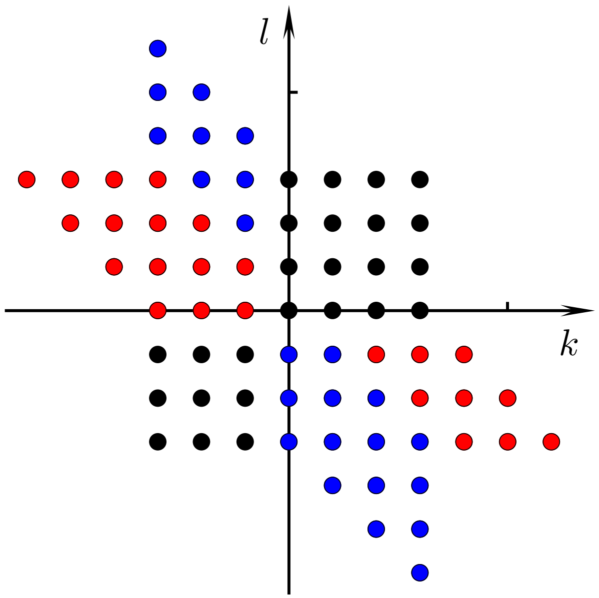
We restrict the indices to $k,l \geq 0$ and $k,l \leq -1$.

$$(k,l) \;\rightarrow\; (l,-k-l) \;\rightarrow\; (-k-l,k)$$

Basis functions on $T^2/\mathbb{Z}_3 = \mathbb{R}^2/\text{p3}$, continued

$$F_{k,l}^{(p)}(z) = \sqrt{3^{-1-\delta_{k,0}\delta_{l,0}}}\,\left[f_{k,l}(z) + p^2\, f_{l,-k-l}(z) + p\, f_{-k-l,k}(z)\right]$$

$$\int \frac{i\,dz\,dz^*}{2(\mathrm{Im}\,\omega)^2}\, F_{k,l}^{(p)}(z)\; F_{m,n}^{(q)*}(z) = \delta_{p,q}\,\delta_{k,m}\,\delta_{l,n}$$



$$F_{k,l}^{(p)}(\omega z) = F_{l,-k-l}^{(p)}(z) = p\, F_{k,l}^{(p)}(z)$$

We restrict the indices to $k, l \geq 0$ and $k, l \leq -1$.

$$(k,l) \;\rightarrow\; (l,-k-l) \;\rightarrow\; (-k-l,k)$$

$$\varphi(z) = \left[\sum_{k,l=-\infty}^{-1} + \sum_{k,l=0}^{\infty}\right] \varphi_{(k,l)}\; F_{k,l}^{(p)}(z)$$

Basis functions on $T^2/\mathbb{Z}_3 = \mathbb{R}^2/\mathrm{p3}$, continued

$$\int \frac{i\,dz\,dz^*}{2(\operatorname{Im}\omega)^2}\,\varphi(z)\,\delta^*(z-z') = \varphi(z')$$

$$\delta(z) = \Big[\sum_{k,l=-\infty}^{-1} + \sum_{k,l=0}^{\infty}\Big]\, d_{k,l}\, F_{k,l}^{(+)}(z)$$

Basis functions on $T^2/\mathbb{Z}_3 = \mathbb{R}^2/\mathrm{p3}$, continued

$$\int \frac{i\,dz\,dz^*}{2(\operatorname{Im}\omega)^2}\,\varphi(z)\;\delta^*(z-z')=\varphi(z')$$

$$\delta(z)=\Big[\sum_{k,l=-\infty}^{-1}+\sum_{k,l=0}^{\infty}\Big]\,F^{(+)*}_{k,l}(0)\;F^{(+)}_{k,l}(z)$$

Basis functions on $T^2/\mathbb{Z}_3 = \mathbb{R}^2/\mathrm{p3}$, continued

$$\int \frac{i\,dz\,dz^*}{2(\mathrm{Im}\,\omega)^2}\,\varphi(z)\,\delta^*(z-z') = \varphi(z')$$

$$\delta(z) = \Big[\sum_{k,l=-\infty}^{-1} + \sum_{k,l=0}^{\infty}\Big]\,F_{k,l}^{(+)*}(0)\,F_{k,l}^{(+)}(z)$$

$$\delta(z) = \delta(\omega z) \neq \delta(-z) \quad \Rightarrow \quad \delta(z-z') \neq \delta(z'-z)$$

Basis functions on $T^2/\mathbb{Z}_3 = \mathbb{R}^2/\text{p3}$, continued

$$\int \frac{i\,dz\,dz^*}{2(\mathrm{Im}\,\omega)^2}\,\varphi(z)\;\delta^*(z-z') = \varphi(z')$$

$$\delta(z) = \Big[\sum_{k,l=-\infty}^{-1} + \sum_{k,l=0}^{\infty}\Big]\,F_{k,l}^{(+)*}(0)\;F_{k,l}^{(+)}(z)$$

$$\delta(z) = \delta(\omega z) \neq \delta(-z) \quad\Rightarrow\quad \delta(z-z') \neq \delta(z'-z)$$

$$\delta(z_1 - z_2) = \sum_{p=+,\,\omega,\,\omega^2}\Big[\sum_{k,l=-\infty}^{-1} + \sum_{k,l=0}^{\infty}\Big]\,F_{k,l}^{(p)}(z_1)\,F_{k,l}^{(p)*}(z_2)$$

The basis functions $F_{k,l}^{(p)}(z)$ are **orthonormal** and **complete**.

Conclusions

- 2 one-dimensional and 17 two-dimensional orbifolds
- distinguish between definition of the orbifold and the physics on it
- consider orbifolds as quotient spaces $\mathbb{R}/\Gamma$ and $\mathbb{R}^2/\Gamma$, rather than S^1/Γ' and T^2/Γ'

  $\Gamma = \langle a, b, c \mid \cdots \rangle \quad \Rightarrow$ parities (α, β, γ)

  Translations are not special! Neither are S^1, T^2 or Scherk-Schwarz phases.
- algebraic definition of space groups important for determination of possible parities
- orthonormality and completeness of the basis essential
- **useful toolkit for model builders**