Flowing to four dimensions

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Outline

1. Motivations
2. The model
3. Six-dimensional description
4. Low energy effective theory
5. The massive spectrum
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1. **Motivations**
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## Motivations

### Phenomenology of X-dim
- Symmetry breaking pattern generating masses $\approx$ TeV
  \[ \Lambda = R^{-1}, \Lambda = M_* \]
- 5D: a bulk or a localised mass term has $\mu^2 = 1$ or 2.
  $\implies$ move to 6D: a localised mass parameter has $\mu^2 = 0$

### Features applicable to QCD in 4D
- Perturbative at UV
- Blowing up in the IR
- Generate light masses compared to the relevant scales

### Theoretical motivations
- Dynamical generation of small masses in a theory which originally had only heavy degrees of freedom
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Action and propagator

**Action**

\[
S = \int d^4 x d^2 y \left[ |\partial_M \phi|^2 - \delta^2(\vec{y}) \cdot \left( -\mu^2 |\phi|^2 + \frac{\lambda}{2} |\phi|^4 \right) \right]
\]

"brane"

**Goldberger and Wise (2001)**

Mixed representation : 4D momentum \((k)\times 2D\) coordinates \((\vec{x})\)

\[
\begin{align*}
\vec{x} & \quad \vec{y} \\
\rightarrow & \quad + \quad + \\
\vec{x} & \quad \mu^2 \\
\vec{y} & \quad \mu^2 \\
\end{align*}
\]

free 6D propagator

\[
G_k^{(2)}(\vec{x}, \vec{y}) = \underbrace{D_k(\vec{x}, \vec{y})} + \frac{\mu^2}{1 - \mu^2 D_k(\vec{0}, \vec{0})} D_k(\vec{x}, \vec{0}) D_k(\vec{0}, \vec{y})
\]

Introduce UV cut-off \(\Lambda \Rightarrow D_k(\vec{0}, \vec{0}) = \frac{1}{4\pi} \ln \frac{\Lambda^2}{k^2}\) for infinite X-dim.
The settings

Compact extra dimensions

Construct the theory on a disk with radius \( R \gg \Lambda^{-1} \). Impose the **Dirichlet boundary condition** \( \phi(r = R) = 0 \). The behaviour of the propagator at the singularity becomes

\[
D_k(\vec{0}, \vec{0}) \sim \frac{1}{4\pi} \ln \frac{\Lambda^2}{k^2} \quad k^2 \gg 1/R^2
\]

\[
D_k(\vec{0}, \vec{0}) \sim \frac{1}{4\pi} \ln \Lambda^2 R^2 + \mathcal{O}(k^2 R^2) \quad k^2 \ll 1/R^2
\]

\[\rightarrow\] Log scale dependence typical of the RG, true at the **classical** level.
**RG flow at the classical level**

### Classical running

- Seen from the resummation
  \[
  \mu^2(Q) = \frac{\mu^2(\Lambda)}{1 + \frac{\mu^2(\Lambda)}{2\pi} \ln \frac{Q}{\Lambda}}
  \]

  \(\mu^2(\Lambda) \equiv \mu^2 \ll 1\) : the coupling grows in the IR.

- If \(\mu^2(R^{-1}) \ll 1 \rightarrow all \ m \sim R^{-1}\).

- What if \(\mu^2(R^{-1}) \rightarrow \infty\)? i.e when
  \[
  \mu^2 = \mu_c^2 = \frac{2\pi}{\ln \Lambda R}.
  \]

Our claim:

- a massless state in the spectrum.

### Fine-tuning \(\mu^2 = \mu_c^2\)

The running picture breaks down at low energies. We should work only with the bare coupling. Identify \(\Lambda = 1/\varepsilon\), \(\varepsilon\) : thickness of the resolved brane.
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**Resolving the brane**

**Regularised potential**

\[
V(\phi) = \frac{1}{\pi \epsilon^2} \left( -\frac{\mu^2}{2} \phi^2 + \frac{\lambda}{4} \phi^4 \right) \quad \text{for } 0 \leq r \leq \epsilon , \\
V(\phi) = 0 \quad \text{for } \epsilon \leq r \leq R .
\]

- Decomposition: \( \phi(r, \theta) = e^{i l \theta} \phi_l(r) , \)
- KK zero mode \( \sim \) s-wave solution \( \phi_0 \equiv \phi , \)
- Need to match the solutions at \( r = \epsilon . \)
Field equations

Equations of motion

\[
\begin{align*}
-p^2 \phi - \Delta_2 \phi + \left[ -\frac{\mu^2}{\pi\epsilon^2} \phi + \frac{\lambda}{\pi\epsilon^2} \phi^3 \right] &= 0 \quad \text{for } 0 \leq r \leq \epsilon , \\
-p^2 \phi - \Delta_2 \phi &= 0 \quad \text{for } \epsilon \leq r \leq R ,
\end{align*}
\]

These are Schrödinger equations in 2D.

General solutions

- Drop the \( \lambda \phi^4 \) term \( \Rightarrow \) background \( \langle \phi \rangle = 0 \). Since \( p^2 \ll \Lambda^2 = \epsilon^{-2} \),

\[
\phi(r) = A \cdot J_0\left( \frac{\mu}{\sqrt{\pi}} \frac{r}{\epsilon} \right) \quad \text{for } \ r \leq \epsilon ,
\]

\[
\phi(r) = B \left[ J_0(|p|r) - \frac{J_0(|p|R)}{N_0(|p|R)} \cdot N_0(|p|r) \right] \quad \text{for } \ r \geq \epsilon ,
\]

when allowing for any sign of \( p^2 \).
Without quartic term

**Light mass state**

Asking for $|p^2| \ll R^{-2}$, one finds

$$p^2 \equiv m^2_{(4)} = \frac{8\pi}{R^2} \frac{\mu_c^2 - \mu^2}{\mu_c^4} = \frac{8\pi}{R^2} \frac{1}{\mu^2(R^{-1})} .$$

- The zero mode is **massless** at the critical coupling. The existence of a massless mode is seen as a pole in the propagator at $p^2 = 0$ and $\mu^2 = \mu_c^2$ since

$$\frac{\mu_c^2}{1 - \mu_c^2D_p(\bar{0}, \bar{0})} \sim \mathcal{O}\left(\frac{1}{p^2R^2}\right) \quad \text{for } p^2 \ll 1/R^2 .$$

- $\mu^2 = \mu_c^2$ is the point of second order phase transition.
Without quartic term

**Light mass state**
- We recover $m^2 \gg R^{-2}$ for $\mu^2(R^{-1}) \ll 1$.
- For $\mu^2(\Lambda) \gtrsim \mu_c^2$, tachyon $\Rightarrow$ consider the $\lambda \phi^4$ term.

**From the boundary condition**
Qualitatively, a Dirichlet BC at $r = R \leftrightarrow$ a Scherk-Schwartz BC.

Naively at least: $m^2_{(4),\text{naive}} = \frac{z_0^2}{R^2} - \frac{\mu^2}{\pi R^2}$
With quartic term

**Symmetry breaking**: $\mu^2 > \mu_c^2$

- The field varies only slightly inside the brane $V'(\phi_c) \approx V'(\phi_0)$ for $r \leq \epsilon$:
  
  $$
  \phi_c(r) = \phi_0 + \frac{1}{4} V'(\phi_0) \cdot r^2 \quad \text{inside},
  $$
  
  $$
  \phi_c(r) = \phi_0 \cdot \frac{\ln R/r}{\ln R/\epsilon} \quad \text{outside}.
  $$

- Matching at $r = \epsilon$: $\phi_0^2 = \frac{\mu^2 - \mu_c^2}{\lambda}.$

- Mass of the fluctuation $\phi(r, p) = \phi_c(r) + \zeta(r, p)$ assuming $m_\zeta^2 \ll R^{-2}$:

  $$
  m_\zeta^2 = \frac{16\pi}{R^2} \cdot \frac{\mu^2 - \mu_c^2}{\mu_c^4} = 2 |m_{\text{tachyon}}^2|
  $$
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Technically

- Decompose $\phi(x^\mu, r) = \sigma(x^\mu) \cdot \zeta(r)$, with $\zeta$ the general solution.
- Impose $\int \zeta^2 d^2y = 1$ so that $\sigma$ has canonical kinetic term.
- Resolve the brane: $\zeta(0) = \zeta(\epsilon)$.

Integration over the X-dim

The 4D effective potential for $\sigma$ is

$$V_{\text{eff}}(\sigma) = \frac{m_{(4)}^2}{2} \sigma^2 + \frac{\lambda_{(4)}}{4} \sigma^4,$$

with $\lambda_{(4)} = \frac{64\pi^2}{\mu_c^8} \frac{\lambda}{R^4}$ and $m_{(4)}^2$ found before.

For $\mu^2 > \mu_c^2$, the 4d field develops a vev $< \sigma^2 > = -\frac{m_{(4)}^2}{\lambda_{(4)}}$
and its fluctuation’s mass is $m_{\zeta}^2 = -2m_{(4)}^2$.
Therefore, the 6D $\leftrightarrow$ 4D equivalence exists in the broken phase.
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KK heavy states

Solutions

• Solve the Schrödinger eq. for \( l \neq 0 \), at \( \mu^2 \leq \mu_c^2 \), above \( \langle \phi \rangle = 0 \) assuming positive squared mass \( p^2 \):

\[
\begin{align*}
\phi_l(r) &= A J_l \left( \sqrt{p^2 + \frac{\mu^2}{\pi \epsilon^2}} r \right) \quad \text{inside}, \\
\phi_l(r) &= B \left[ J_l(pr) - \frac{J_l(pR)}{N_l(pR)} N_l(pr) \right] \quad \text{outside}.
\end{align*}
\]

Matching at \( r = \epsilon \), with \( M \epsilon \ll 1 \) and \( \mu^2 \ll 1 \)

\[
\frac{2}{\pi} \frac{J_0(MR)}{N_0(MR)} \cdot \left\{ \ln \frac{MR}{2} + \frac{\mu_c^2 - \mu^2}{\mu_c^2} \ln \Lambda R \right\} = 1.
\]

At the critical coupling, the KK masses have no dependence on \( \Lambda \)
The return of the running

Expression at the pole

\[ \frac{J_0(MR)}{N_0(MR)} = \frac{1}{4} \mu^2 (Q = M) \]

See also:

\[ \frac{\mu_c^2}{1 - \mu_c^2 D_p(\bar{0}, \bar{0})} = \frac{2\pi}{\ln(pR)} \quad \text{for } p^2 \gg 1/R^2. \]

Precisely at the critical coupling, all physical quantities become independent on the UV cut-off.
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In a nutshell

Open questions and prospects

- A new class of models which provide a new way to get small masses. Possible application to electroweak or supersymmetry breaking?
- Similar light modes for chiral fermions? \( \Rightarrow \) yes
- Incorporate gravity
- How to drive parameters near a critical point? What would be the dynamics or the symmetries of such a mechanism?
- Are such models dual to Nambu-Jona-Lasinio theories? \( \Rightarrow \) work in progress