NILPOTENT SPINOR SYMMETRY with INTERACTING SPIN 3/2 FIELD *)

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(I) Non-Abelian Tensor with Consistent Interactions (Bosonic)

(II) Fermionic Nilpotent Symmetry

(III) Conclusions

*) The title has been slightly changed, but the content is the same.

(I) Non-Abelian Tensor (Bosonic Case)

(i) Problem with NA-Tensor

$B_{\mu\nu}{}^I$: 2-nd Rank in Adj. Rep.

$A_\mu{}^I$: Usual Gauge Field

Field Strength for $B_{\mu\nu}{}^I$:

$$G_{\mu\nu\rho}{}^I \equiv 3 D_{[\mu} B_{\nu\rho]}{}^I$$
$$= 3 \left[\partial_{[\mu} B_{\nu\rho]}{}^I + f^{IJK} A_{[\mu}{}^J B_{\nu\rho]}{}^K \right]$$

A Lagrangian:

$$\mathcal{L}_0 = -\frac{1}{12} (G_{\mu\nu\rho}{}^I)^2 - \frac{1}{4} (F_{\mu\nu}{}^I)^2$$

Field Eq. of $B_{\mu\nu}{}^I$:

$$\frac{\delta \mathcal{L}_0}{\delta B_{\mu\nu}{}^I} = + \frac{1}{2} D_\rho G^{\mu\nu\rho I} \doteq 0 .$$

$$0 \overset{?}{=} D_\mu \left(\frac{\delta \mathcal{L}_0}{\delta B_{\mu\nu}{}^I} \right) = \frac{1}{4} f^{IJK} F_{\mu\rho}{}^J G^{\mu\nu\rho K} \neq 0$$

Another Problem:

$$\delta_\Lambda B_{\mu\nu}{}^I = 2 D_{[\mu} \Lambda_{\nu]}{}^I ,$$
$$\delta_\Lambda G_{\mu\nu}{}^I = +3 f^{IJK} F_{[\mu\nu}{}^J \Lambda_{\rho]}{}^K \neq 0$$
$$\Longrightarrow \delta_\Lambda I_0 \neq 0$$

A Clue to solve this problem can be found in Scherk-Schwarz Dimensional Reduction:

$$D + \underset{\substack{\uparrow \\ \text{Extra Dims.}}}{E} \longrightarrow D$$

$$(\hat{x}^{\hat{\mu}}) = (x^\mu, y^\alpha)$$

$$\mu = 0, 1, \cdots, D-1; \quad \alpha = 1, 2, \cdots, E$$

$$\hat{G}_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}} \rightarrow \begin{cases} G_{\mu\nu\rho\sigma} = 4\partial_{[\mu} B_{\nu\rho\sigma]} + 10 F_{[\mu\nu}{}^\alpha B_{\rho\sigma]\alpha}, \\ G_{\mu\nu\rho\alpha} = 3\partial_{[\mu} B_{\nu\rho]\alpha} + 6 F_{[\mu\nu}{}^\beta B_{\rho]\alpha\beta}, \\ G_{\mu\nu\alpha\beta} = 2\partial_{[\mu} B_{\nu]\alpha\beta} + F_{\mu\nu}{}^\gamma B_{\alpha\beta\gamma} \\ \qquad\qquad - f_{\alpha\beta}{}^\gamma B_{\mu\nu\gamma}, \\ G_{\mu\alpha\beta\gamma} = D_\mu B_{\alpha\beta\gamma} + 3 f_{[\alpha\beta|}{}^\delta B_{\mu\delta|\gamma]}, \\ G_{\alpha\beta\gamma\delta} = -6 f_{[\alpha\beta|}{}^\epsilon B_{\epsilon|\gamma\delta]}. \end{cases}$$

$$\uparrow$$

$$\hat{B}_{\hat{\mu}\hat{\nu}\hat{\rho}}$$

$f_{\alpha\beta}{}^\gamma$: Structure Const. for "Flat Group" to give masses to fields in D dims.

$\Rightarrow$ Mimic this structure for NA Tensors!

$\Rightarrow$ Replace $\alpha, \beta, \ldots$ indices by Adjoint indices $I, J, \ldots$!

(ii) Simple Ex. of Consistent NA Tensor

Introduce the Set of Fields:

$$(B_{\mu\nu}{}^{I},\ C_{\mu}{}^{IJ};\ K^{IJK};\ A_{\mu}{}^{I})$$

and their Field Strengths:

$$\begin{cases} G_{\mu\nu\rho}{}^{I} \equiv +3D_{[\mu}B_{\nu\rho]}{}^{I} - 3F_{[\mu\nu}{}^{J}C_{\rho]}{}^{JI}, \\ H_{\mu\nu}{}^{IJ} \equiv +2D_{[\mu}C_{\nu]}{}^{IJ} + F_{\mu\nu}{}^{K}K^{IJK} + f^{IJK}B_{\mu\nu}{}^{K}, \\ L_{\mu}{}^{IJK} \equiv D_{\mu}K^{IJK} - 3f^{[IJ|L}C_{\mu}{}^{L|K]}, \end{cases}$$

invariant under Λ-Trsfs:

$$\begin{cases} \delta_\Lambda B_{\mu\nu}{}^{I} = 2D_{[\mu}\Lambda_{\nu]}{}^{I} - F_{\mu\nu}{}^{J}\Lambda^{IJ}, \\ \delta_\Lambda C_{\mu}{}^{IJ} = D_{\mu}\Lambda^{IJ} - f^{IJK}\Lambda_{\mu}{}^{K}, \\ \delta_\Lambda K^{IJK} = 3f^{[IJ|L}\Lambda^{L|K]}, \\ \delta_\Lambda A_{\mu}{}^{I} = 0 \end{cases}$$

$$\Longrightarrow \delta_\Lambda G_{\mu\nu\rho}{}^{I} = 0,\ \delta_\Lambda H_{\mu\nu}{}^{IJ} = 0,\ \delta_\Lambda L_{\mu}{}^{IJK} = 0.$$

Field Strengths are Λ-Transformation Invariant!

Bianchi Ids:

$$D_{[\mu} G_{\nu\rho\sigma]}{}^{I} \equiv + \frac{3}{2} F_{[\mu\nu}{}^{J} H_{\rho\sigma]}{}^{IJ} ,$$

$$D_{[\mu} H_{\nu\rho]}{}^{IJ} \equiv + F_{[\mu\nu}{}^{K} L_{\rho]}{}^{IJK} + \frac{1}{3} f^{IJK} G_{\mu\nu\rho}{}^{K} ,$$

$$D_{[\mu} L_{\nu]}{}^{IJK} \equiv +3 f^{[IJ|L} F_{\mu\nu}{}^{M} K^{L|KM]} - \frac{3}{2} f^{[IJ|L} H_{\mu\nu}{}^{L|K]} .$$

(Generalized Chern-Simons Terms)

A Typical Invariant Action in $\forall D$:

$$I_1 \equiv \int d^D x \, \mathcal{L}_1 = \int d^D x \left[-\frac{1}{12} (G_{\mu\nu\rho}{}^{I})^2 - \frac{1}{4} (H_{\mu\nu}{}^{IJ})^2 - \frac{1}{2} (L_{\mu}{}^{IJK})^2 - \frac{1}{4} (F_{\mu\nu}{}^{I})^2 \right] ,$$

Field Eqs.:

$$\frac{\delta \mathcal{L}_1}{\delta B_{\mu\nu}{}^{I}} = + \frac{1}{2} D_\rho G^{\mu\nu\rho I} - \frac{1}{2} f^{IJK} H^{\mu\nu JK} \doteq 0 ,$$

$$\frac{\delta \mathcal{L}_1}{\delta C_{\mu}{}^{IJ}} = - D_\nu H^{\mu\nu IJ} + \frac{1}{2} F_{\nu\rho}{}^{[I|} G^{\mu\nu\rho|J]} - 3 f^{[I|KL} L^{\mu KL|J]} \doteq 0 ,$$

$$\frac{\delta \mathcal{L}_1}{\delta K^{IJK}} = D_\mu L^{\mu IJK} - \frac{1}{2} F^{\mu\nu[I} H_{\mu\nu}{}^{JK]} \doteq 0 ,$$

$$\frac{\delta \mathcal{L}_1}{\delta A_{\mu}{}^{I}} = - D_\nu F^{\mu\nu I} + \frac{1}{2} f^{IJK} G^{\nu\rho\sigma J} B_{\rho\sigma}{}^{K} + D_\rho (C_\sigma{}^{IJ} G^{\mu\rho\sigma J}) + 2 f^{IJL} H^{\mu\nu LK} C_\nu{}^{JK} - D_\nu (K^{IJK} H^{\mu\nu JK}) + 3 f^{IJK} K^{KLM} L^{\mu JLM} \doteq 0 .$$

Consistencies:

$$D_\nu\left(\frac{\delta \mathcal{L}_1}{\delta B_{\mu\nu}{}^I}\right) \doteq 0, \quad D_\mu\left(\frac{\delta \mathcal{L}_1}{\delta C_\mu{}^{IJ}}\right) \doteq 0, \quad D_\mu\left(\frac{\delta \mathcal{L}_1}{\delta A_\mu{}^I}\right) \doteq 0.$$

Mass Spectrum:

☆ The $B_{\mu\nu}$-Field Eq. is Massive KG:

$$\frac{\delta \mathcal{L}_1}{\delta B_{\mu\nu}{}^I} = +\frac{1}{2}\left(D_\rho D^\rho \hat{B}^{\mu\nu I} - \underbrace{a_0}_{\text{Mass } m_{\hat{B}}{}^2} B^{\mu\nu I}\right) + \mathcal{O}(\phi^2) \doteq 0$$

$$f^{IJK} f^{JKL} \equiv a_0 \delta^{IL} \quad (a_0 > 0)$$

$$\hat{B}_{\mu\nu}{}^I \equiv B_{\mu\nu}{}^I + 2a_0^{-1} f^{IJK} D_{[\mu} C_{\nu]}{}^{JK},$$

Gauge Cond.: $D_\mu \hat{B}^{\mu\nu} \overset{*}{=} 0.$

Mass^2: $m_{\hat{B}}^2 = a_0 > 0$ (Non-Tachyonic)

☆ The C_μ-Field Eq. is also Massive:

$$\frac{\delta \mathcal{L}_1}{\delta C_\mu{}^{IJ}} = P^{IJ,KL}\left(D_\nu D^\nu \hat{C}_\mu{}^{KL} - 3a_0 \hat{C}_\mu{}^{KL}\right) + \mathcal{O}(\phi^2) \doteq 0,$$

$$\hat{C}_\mu{}^{IJ} \equiv P^{IJ,KL} C_\mu{}^{KL} - a_0^{-1} f^{[I|KL} D_\mu K^{KL|J]},$$

Gauge Cond.: $D_\mu \hat{C}^{\mu IJ} \overset{*}{=} 0.$

The P's are Projection Operators:

$$P^{IJ,KL} \equiv \delta^{[I|K} \delta^{|J]L} - a_0 h^{IJ,KL},$$

$$h^{IJ,KL} \equiv f^{IJM} f^{MKL} \equiv a_0 Q^{IJ,KL},$$

$$P^{IJ,KL}P^{KL,MN} = P^{IJ,MN}, \quad Q^{IJ,KL}Q^{KL,MN} = Q^{IJ,MN}$$

$$P^{IJ,KL} + Q^{IJ,KL} = \delta^{[I|K}\delta^{|J]L},$$

$$P^{IJ,KL}Q^{KL,MN} = 0, \quad Q^{IJ,KL}P^{KL,MN} = 0.$$

The C-field eq. above implies that only P-Projected components of C satisfy the massive KG eq. with $m_C^2 \equiv 3a_0 > 0$ (Non-Tachyonic).

P	Q	Total
$\frac{g(g-3)}{2}$	g	$\frac{g(g-1)}{2}$

↑ Auxiliary (decoupled from System)

The K-field eq. is massless KG:

$$\frac{\delta \mathcal{L}_1}{\delta K^{IJK}} = D_\mu D^\mu \hat{K}^{IJK} + \mathcal{O}(\phi^2) \doteq 0$$

$$\hat{K}^{IJK} \equiv K^{IJK} - \frac{3}{2}a_0^{-1} h^{[IJ|MN} K^{MN|K]} + \frac{3}{2}a_0 f^{[IJ|L} f^{|K]MN} K^{MNL}$$

(II) Fermionic Nilpotent Symmetry

We can mimic the bosonic results for fermionic system for $\forall D$.

(i) Algebra & Field Representations

$$\{Q_\alpha^I, Q_\beta^J\} = 0, \quad \left(\begin{array}{l} I, J, \cdots \text{ adjoint} \\ \alpha, \beta, \cdots \text{ Majorana Spinor} \end{array}\right)$$

$$[T^I, Q_\alpha^J] = +f^{IJK} Q_\alpha^K,$$

$$[T^I, T^J] = +f^{IJK} T^K.$$

Set of Fields:

Previous Bosonic $\longleftrightarrow$ Present Fermionic

$$(B_{\mu\nu}^I, C_\mu^{IJ}, K^{IJK}, A_\mu^I) \longleftrightarrow (\psi_{\mu\alpha}^I, \chi_\alpha^{IJ}, A_\mu^I)$$

Potentials $\longleftrightarrow$ Field Strengths

$$\psi_\mu^\alpha \longleftrightarrow R_{\mu\nu}^\alpha \equiv 2 D_{[\mu}\psi_{\nu]}^I + F_{\mu\nu}^J \chi^{IJ},$$

$$\chi_\alpha^{IJ} \longleftrightarrow L_\mu^{IJ} \equiv D_\mu \chi^{IJ} + f^{IJK} \psi_\mu^K,$$

$$A_\mu^I \longleftrightarrow F_{\mu\nu}^I \equiv 2\partial_{[\mu} A_{\nu]}^I + \underbrace{f^{IJK} A_\mu^J A_\nu^K}_{\text{Generalized CS}}.$$

(ii) Nilpotent Fermionic Sym.:

$$\begin{cases} \delta_Q \psi_\mu{}^I = D_\mu \varepsilon^I, \quad \delta_Q A_\mu{}^I = 0, \\ \delta_Q \chi^{IJ} = -f^{IJK} \varepsilon^K = -Q^{IJ,KL} f^{KLM} \varepsilon^M, \end{cases}$$

$$\therefore \delta_Q \mathcal{R}_{\mu\nu}{}^I = 0, \quad \delta_Q L_\mu{}^{IJ} = 0.$$

Non-Abelian Gauge Trsf.:

$$\begin{cases} \delta_T A_\mu{}^I = D_\mu \alpha^I, \\ \delta_T \psi_\mu{}^I = -f^{IJK} \alpha^J \psi_\mu{}^K, \\ \delta_T \chi^{IJ} = -2 f^{[I|KL} \alpha^K \chi^{L|J]}, \end{cases}$$

$$\therefore \begin{cases} \delta_T \mathcal{R}_{\mu\nu}{}^I = -2 f^{IJK} \alpha^J \mathcal{R}_{\mu\nu}{}^K, \\ \delta_T L_\mu{}^{IJ} = -2 f^{[I|KL} \alpha^K L_\mu{}^{L|J]}. \end{cases}$$

(iii) Total Action: $I_T = I_2 + I_3 + I_4$:

$$I_2 \equiv \int d^D x \, \mathcal{L}_2 = \int d^D x \left[+ \tfrac{1}{4} a_0^{-1} f^{IJK} (\overline{L}_\mu{}^{IJ} \gamma^{\mu\nu\rho} \mathcal{R}_{\nu\rho}{}^K) \right]$$

$$= \int d^D x \left[\underbrace{+ \tfrac{1}{4} (\overline{\psi}_\mu{}^I \gamma^{\mu\nu\rho} \mathcal{R}_{\nu\rho}{}^I)}_{\text{Rarita-Schwinger}} - \tfrac{1}{4} a_0^{-1} f^{IJK} (\overline{\chi}^{IJ} \gamma^{\mu\nu\rho} L_\mu{}^{KL}) F_{\nu\rho}{}^L \right]$$

$$I_3 \equiv \int d^D x \, \mathcal{L}_3 = \int d^D x \left[+ \tfrac{1}{2} P^{IJ,KL} (\overline{\chi}^{IJ} \gamma^\mu L_\mu{}^{KL}) \right]$$

$$= \int d^D x \left[\underbrace{+ \tfrac{1}{2} P^{IJ,KL} (\overline{\chi}^{IJ} \gamma^\mu D_\mu \chi^{KL})}_{\text{Kinetic Term of } \chi} \right],$$

$$I_4 \equiv \int d^D x \, \mathcal{L}_4 = \int d^D x \left[-\tfrac{1}{4} (F_{\mu\nu}{}^I)^2 \right].$$

(iv) Consistency of Field Eqs.

$$\frac{\delta \mathcal{L}_T}{\delta \psi_\mu{}^I} = \frac{1}{2}\gamma^{\mu\rho\sigma}\mathcal{R}_{\rho\sigma}{}^I - \frac{1}{4}\gamma^{\mu\rho\sigma}\chi^{IJ}F_{\rho\sigma}{}^J + \frac{1}{4}Q^{IJ,KL}\gamma^{\mu\rho\sigma}\chi^{KL}F_{\rho\sigma}{}^J,$$

$$D_\mu\left(\frac{\delta \mathcal{L}_T}{\delta \psi_\mu{}^I}\right) = 0 \ ?$$

In fact, we have the Identity:

$$D_\mu\left(\frac{\delta \mathcal{L}_T}{\delta \psi_\mu{}^I}\right) + f^{IJK}\left(\frac{\delta \mathcal{L}_T}{\delta \chi^{JK}}\right) \equiv 0,$$

due to the invariance $\delta_\alpha I_T = 0$.

This is the consequence of the $F\chi$-term or ψ-lin. term in $\mathcal{R}$ or L-field strength!

(V) Quantization

☆ BRST-Invariance

$$I'_{tot} = I_T + I_{GF} + I_{FP},$$

$$\mathcal{L}_{GF} \equiv \frac{1}{4}(\bar{\psi}_\mu{}^I \gamma^\mu \partial\!\!\!/ \gamma^\nu \psi_\nu{}^I) + \frac{1}{2}(\partial_\mu A^{\mu I})^2,$$

$$\mathcal{L}_{FP} \equiv +(\xi^{*I} D\!\!\!\!/\, \xi^I) - f^{IJK}(\xi^{*I}\gamma^\mu \psi_\mu{}^J) C^K - (\partial_\mu C^{*I})(D^\mu C^I),$$

	Ghosts	Anti-Ghosts
Anti-Comm	C^I	C^{*I}
Comm	$\xi^{\alpha I}$	$\xi^{*\alpha I}$

☆ BRST Trsfs.

$$\delta_B \psi_\mu{}^I = (D_\mu \xi^I)\lambda - f^{IJK} C^J \psi_\mu{}^K \lambda$$

$$\delta_B A_\mu{}^I = -(D_\mu C^I)\lambda,$$

$$\delta_B \chi^{IJ} = -f^{IJK}\xi^K \lambda - 2 f^{[I|LM} C^L \chi^{M|J]}\lambda$$

$$\delta_B \xi^{\alpha I} = +f^{IJK}\xi^{\alpha J} C^K \lambda,$$

$$\delta_B C^I = -\frac{1}{2} f^{IJK} C^J C^K \lambda,$$

$$\delta_B \xi^{*\alpha I} = +\frac{1}{2}(\partial\!\!\!/ \gamma^\mu \psi_\mu{}^I)^\alpha \lambda,$$

$$\delta_B C^{*I} = -(\partial_\mu A^{\mu I})\lambda,$$

$$\lambda^2 = 0 \quad (\text{Nilpotent}),$$

$$\Longrightarrow \delta_B I_{TOT} = 0.$$

(III) Conclusions

We have consistent interactions for Spin 3/2 Field $\psi_\mu{}^{\alpha I}$ both with spinorial and adjoint indices!

The associated Field Strengths are inspired by Non-Abelian Tensor system.

Similarly to NA-Tensor Formulation, our Field strength $\mathcal{R}_{\mu\nu}{}^{\alpha I}$ for $\psi_\mu{}^{\alpha I}$ contains generalized Chern-Simons terms.

We seem to be on the 'Right Track' for formulating Interacting Spin 3/2 fields.

We gave first non-trivial Interacting model for Nilpotent Fermionic Generators $\{Q_\alpha{}^I, Q_\beta{}^J\} = 0$ for $\forall D$.