The Recursive Relation between the Field Equation and the Quantization of the Field

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Introduction

Quantization relations and constraints

- By using Lagrangian and quantization of a field, most meaningful quantities, if not all, can be obtained.
- It was pointed out, by Yeong-Shyeong Tsai *et al.* [1], that the quantization relations of field variables are nothing but **constraints** of the field variables.
- Therefore, the field equations must be derived from the Lagrangian by applying the method of **Lagrange multipliers**.

Introduction

Fourier Expansion and Feynman Diagrams

- Clearly, the quantization relations must be settled before the field equation are derived.
- Therefore, it is inappropriate to say that the quantization relations are derived from the Fourier expansion of field variables.
- Some approach might works very well without using the Fourier expansions of field variables.
- But, as we know, there will be neither particle creations nor particle annihilations without the Fourier expansions of field variables.

- Then the pictorial representations of perturbation theory, the picture of particle productions and particle destructions, are fictitious and unrealistic.
- Actually, Feynman diagrams, the pictorial representations of particle creations and destructions, are indispensable in QFT and the experimental data of high energy physics have shown us that there are many events of particle creations and destructions in the real world.
- It is the Fourier expansion of field variables that connects the abstract QFT with nature phenomena of particle physics.

Introduction

The prior quantization and the field equation

- If the plane wave solutions of the field equations can be obtained, then there will be the Fourier expansion of the field variables and the relation between field variables, including all the quantization, can be recalculated.
- Consequently, there are two versions of quantization relations. One is **prior** to the field equations, and the other is **posterior** to the field equations.
- They are called **prior quantization** and **posterior quantization** respectively.

- Of course, the posterior quantization should be **the same** as the prior one.
- Hence, there is **a strong relation** between the prior quantization and the field equation of a field, especially the fields of spin 1/2.
- Clearly, this relation is **recursive**, and hence it is called the recursive relation between quantization and the field equation.

Introduction

The distribution of operators and the thermal factors

- It was also pointed out, by Hung-Ming Tsai *et al.* [1], that QFT treats **a system of particles** rather than a single particle.
- In speaking of a system of particles, **the stochastic behavior** of particles must be taken into consideration.
- In order to fit this fact, a proper **distribution function** or **thermal factor** must be introduced in the mathematical expression of field variables which are associated with particle creations and particle destructions.

- Therefore, the posterior quantization shall be changed, and thus the prior quantization must be modified.
- The recursive relation mentioned above makes QFT more complex and more difficult to handle.
- In this paper, it is shown how to handle or solve this complicated problem.
- In order to obtain a complete and consistent theory, **the fuzzy theory** is introduced into QFT.
- **QED**, the interacting Dirac field, is taken as a **template** in the following discussion.

Quantization, constraints and Lagrange multipliers

The method of Largrange multiplier and field equation

- In order to make it simpler, some of the results of the work by Y. -S. Tsai *et al.* [1] are **quoted** here.

[1] Yeong-Shyeong Tsai, Hung-Ming Tsai, Po-Yu Tsai and Lu-Hsing Tsai, "*The solution of the Dirac equation with the interaction term*," in Proceedings of the 3rd International Symposium on Quantum Theory and Symmetries (QTS3), Cincinnati, Ohio, 10-14 September 2003, World Scientific, Singapore (2004), pp. 431-436.

By applying the method of **Lagrange multiplier** to the conventional Lagrangian and the conventional quantization relations of the interacting Dirac field, **the field equations** with some **parameters**, the Lagrange multipliers, are obtained

$$i\gamma^{\mu}\partial_{\mu}\Psi - m\Psi + \Lambda\Psi = q\gamma^{\mu}\Psi A_{\mu}, \quad (1)$$

where $\Lambda = \begin{bmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} & \lambda_{14} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} & \lambda_{24} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} & \lambda_{34} \\ \lambda_{41} & \lambda_{42} & \lambda_{43} & \lambda_{44} \end{bmatrix}$

and λ_{ij} $i = 1, 2, \cdots, 4,$ $j = 1, 2, \cdots, 4$ are going to be determined.

- Multiply both sides of the equation by $\overline{\Psi}(\mathbf{x}',t)$ from the left, multiply both sides of the equation by $\overline{\Psi}(\mathbf{x}',t)$ from the right and take the sum of them. Then take the three-dimensional integration, the variable of the integration is $\mathbf{x}'$. We obtain

$$\gamma^{\mu}\partial_{\mu}\Psi + U\Psi = 0, \qquad (2)$$

where $U_{\alpha\beta} = \int_{\mathbf{x}'^3} \{\overline{\Psi}_{\beta}, (\gamma^{\mu}\partial_{\mu}\Psi)_{\alpha}\} d\mathbf{x}'$.

- Since $\overline{\Psi}$ and Ψ are dependent, Eq. (2) is a **highly nonlinear** partial differential integral equation.

• Though it is very **difficult** to solve this equation, **the solution set** of equation (2) contains **the solution space** of equation (3).

$$\gamma^{\mu}\partial_{\mu}\Psi = 0. \qquad (3)$$

• If we limit the solution of Eq. (2) to be a member of the solution space of Eq. (3), there will be **Fourier expansion** because Eq. (3) is **linear**.

$$\Psi^{res} = \sum_{\mathbf{p}}\sum_{s} N(b_p^s u^s(p)\exp\theta + d_{\mathbf{p}}^{s\dagger} v^s(p)\exp(-\theta)), \qquad (4)$$

where $\theta = i(\mathbf{p}\cdot\mathbf{x} - t|\mathbf{p}|)$, and $|z| = \sqrt{z_1^2 + z_2^2 + z_3^2}$.

- Therefore, we have **operators** of particle production and destruction.
- These procedures and results are the **cornerstones** of later developments.

The Relative Frequency of events

- Most statisticians agree that the probabilities are **relative frequency of events**.
- Here, the **ratios** of occurrences among the events **instead of** the exact number of occurrences are emphasized.
- Let us review simple mathematics in quantum mechanics.

- Consider a simple mathematical expression of quantum state vector

$$|\phi\rangle = c_+ |+z\rangle + c_- |-z\rangle$$

where $|\phi\rangle$ is the state of an electron which is a linear combination of $|+z\rangle$ and $|-z\rangle$, where $|+z\rangle$ is the spin up state and $|-z\rangle$ is spin down state vector along the z-axis direction, the orientation of SG device.

- $|c_+|^2$ and $|c_-|^2$ are interpreted **the probabilities** that the electron in the state $|\phi\rangle$ will be found to be in the states $|+z\rangle$ and $|-z\rangle$ respectively.

- Though this is a **simple mathematical expression**, it can **predict** all the results of experiments about measuring the spin of electrons.

- Since this idea is simple and successful, we

extend this simple idea to the **Fourier expansion** of field variable. Therefore, we have

$$\Psi = \sum_i \sum_{\mathbf{p},(\varepsilon_i - \frac{\Delta\varepsilon}{2} \le E \le \varepsilon_i + \frac{\Delta\varepsilon}{2})} \sum_s (\xi_p^s b_{\mathbf{p}}^s + \zeta_p^s d_{\mathbf{p}}^{s\dagger}) \qquad (5)$$

- Then the length of the vectors, $\overline{\xi}_p^s \xi_p^s$ and $\overline{\zeta}_p^s \zeta_p^s$,

must be associated with **the distribution functions**

$$\frac{1}{1+\exp\frac{(E-\mu)}{k_B T}} \quad \text{and} \quad \frac{1}{\exp\frac{(E-\mu)}{k_B T} - 1} \quad \text{for ferminons}$$

and bosons respectively.

The allocation of operators in a field variable

- Suppose that there are M particles to be distributed into all possible quantum states.
- From (5), we define **a set of quantum states**

$$S_i = \{q_{\mathbf{p}}; \varepsilon_i - \frac{\Delta\varepsilon}{2} \le \sqrt{m^2 + |\mathbf{p}|^2} \le \varepsilon_i + \frac{\Delta\varepsilon}{2}\},$$

where $q_{\mathbf{p}}$ is **a quantum state** of a particle with momentum $\mathbf{p}$.

- Let g_i be the **number** of elements of this set.
- It is obvious that $g_i = 8\pi|\mathbf{p}_i|^2 \Delta|\mathbf{p}_i|$, where

$4\pi|\mathbf{p}_i|^2 \Delta|\mathbf{p}_i|$ is the volume of the spherical shell with radius $|\mathbf{p}|$, 2 is two kinds of spin, $|\mathbf{p}_i| = \sqrt{\varepsilon_i^2 - m^2}$ and $\Delta|\mathbf{p}_i| = \dfrac{\varepsilon_i}{\sqrt{\varepsilon_i^2 - m^2}} \Delta\varepsilon$.

- Clearly, there should be $$\frac{\alpha g_i}{1+\exp\frac{(\varepsilon_i-\mu)}{k_B T}}$$

number of particle in this set [3].

- Since we are interested in the ratio of numbers,

we set $\alpha = 1$.

[3] A. Yariv, *An Introduction to Theory and Application of Quantum Mechanics,* John Wiley & Son, (1982).

- In this set, all $q_{\mathbf{p}}$ are treated **equally likely,** and hence the **probability** of each $q_{\mathbf{p}}$ to accept a particle is $\frac{1}{g_i}$ since there are g_i number of $q_{\mathbf{p}}$.
- Therefore,

$$\overline{\xi}_p^s \xi_p^s = \frac{1}{1 + \exp \frac{(\sqrt{|\mathbf{p}|^2 + m^2} - \mu)}{k_B T}}.$$

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- Since ξ_p^s is the product of scalar factor and an unit vector, the **thermal factor** is

$$\sqrt{\frac{1}{1+\exp\frac{(\sqrt{|\mathbf{p}|^2+m^2}-\mu)}{k_B T}}}.$$

- We get an important result that **the absolute value of the coefficients** of $b_{\mathbf{p}}^s$, $b_{\mathbf{p}}^{s\dagger}$, $d_{\mathbf{p}}^s$ and $d_{\mathbf{p}}^{s\dagger}$ in Fourier expansion **can not** be functions of momentum $\mathbf{p}$ **except** the thermal factors.

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- Furthermore, the analysis of distribution function is based on that **the state vector** on which the field operator, Ψ, operates **must be so nice** that **the distribution of particles** is **Fermi-Dirac** (Bose—Einstein) or it is **a state without particles**.
- Otherwise the thermal factors do not make sense any longer.
- Ideally, **the state** that a field operator with thermal factors can operate is **the vacuum state** $|0\rangle$.

In order to standardize the above requirements, we have **two rules**:

Rule (I)

The absolute value of the **coefficients** of the **operators** $b_{\mathbf{p}}^{s}$, $b_{\mathbf{p}}^{s\dagger}$, $d_{\mathbf{p}}^{s}$ and $d_{\mathbf{p}}^{s\dagger}$ in the Fourier expansion of a field variable **can not** be the function of momentum $\mathbf{p}$ explicitly or implicitly **except** the **thermal factors**.

Rule (II)

If **the whole field operator** Ψ $(\overline{\Psi})$, but **not** a part of the field operator Ψ $(\overline{\Psi})$ such as $\Psi b_{\mathbf{p}}^{s\dagger}$, $\{\Psi, b_{\mathbf{p}}^{s\dagger}\}$ and $b_{\mathbf{p}}^{s\dagger}$, **operates** on the **vacuum state** $|0\rangle$, then the **thermal factor** must be included.

The reason why we must have the **Rule (II)** is that **the field operator** Ψ $(\overline{\Psi})$, the collection of operators of all possible states without any bias, **is associated with the whole system of particles without any exception** hence **the distribution must be considered**, but for **a portion of field operator** Ψ $(\overline{\Psi})$ such as $\Psi b_{\mathbf{p}}^{s\dagger}$, $\{\Psi, b_{\mathbf{p}}^{s\dagger}\}$ and $b_{\mathbf{p}}^{s\dagger}$, the distribution does not make any sense because they are **biased purposely** by a particular selection rule.

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The field variable with the thermal factors

- Before the new mathematics is introduced, the field variables is **assumed temporarily** to be

$$\Psi = \sum_{\mathbf{p}} \sum_{s} \frac{N}{\sqrt{V}} \sqrt{\frac{1}{1+\exp\dfrac{(E-\mu)}{k_B T}}} \sqrt{\frac{m}{E}} (b_{\mathbf{p}}^s u^s(p)\exp\theta + d_{\mathbf{p}}^{s\dagger} v^s(p)\exp(-\theta))$$

where $\theta = i(\mathbf{p}\cdot\mathbf{x} - Et)$, $E = \sqrt{|\mathbf{p}|^2 + m^2}$, N is a factor for normalization, T is the temperature , μ is the chemical potential and k_B is a the Boltzmann constant.

The posterior quantization

- Hence

$$\{\Psi_{\alpha}(\mathbf{x},t),\overline{\Psi}_{\beta}(\mathbf{x}',t')\}=(i\gamma^{\mu}\partial_{\mu}+m)_{\alpha\beta}\left(-\frac{N}{(2\pi)^{3}}\int_{\mathbf{p}^{3}}\frac{e^{ip\cdot(x-x')}-e^{-ip\cdot(x-x')}}{2E}\frac{1}{(1+\exp\frac{(E-\mu)}{k_{B}T})}d\mathbf{p}\right)$$

(7)

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Via posterior quantization to prior quantization

Prior quantization with thermal factors

- It is assumed that the prior quantization of interacting Dirac field is quite similar to or the same as Eq. (7) though it is the posterior quantization originally.

- The quantization is defined by a **first-order differential equation** with **initial condition**, so to speak. Let

$$S^{M}_{\rho\sigma}(\mathbf{x}-\mathbf{x}',t-t')=\{\Psi_{\rho}(\mathbf{x},t),\overline{\Psi}_{\sigma}(\mathbf{x}',t')\}.$$

$$(i\gamma^{\mu}\partial_{\mu}-m)_{\alpha\gamma}S^{M}_{\gamma\beta}(\mathbf{x}-\mathbf{x}',t-t')=0,$$

$$S^{M}_{\rho\sigma}(0,0)=\delta_{\rho\sigma}a \quad \text{and} \quad a>0. \qquad (8)$$

- By applying the method of **Lagrange multiplier**, the field equation, which is the same as (1) is obtained

$$i\gamma^{\mu}\partial_{\mu}\Psi-m\Psi+\Lambda\Psi=q\gamma^{\mu}\Psi A_{\mu}. \qquad (1)$$

and the Maxwell equation is

$$\Box A=q\overline{\Psi}\gamma^{\mu}\Psi.$$

Multiply **both sides** of the α^{th} equation in (1) by $\overline{\Psi}_{\beta}(\mathbf{x}',t')$ **from the right**, do the same procedure but multiplying **from the left** and take their **sum**. We obtain

$$(i\gamma^{\mu}\partial_{\mu} - m + \Lambda)_{\alpha\gamma} S^{M}_{\gamma\beta} = (q\gamma^{\mu}A_{\mu})_{\alpha\gamma} S^{M}_{\gamma\beta}$$

Since

$$(i\gamma^{\mu}\partial_{\mu} - m)_{\alpha\gamma} S^{M}_{\gamma\beta}(\mathbf{x}-\mathbf{x}',t-t') = 0$$

we get

$$(m - m + \Lambda)_{\alpha\gamma} S^{M}_{\gamma\beta} = (q\gamma^{\mu}A_{\mu})_{\alpha\gamma} S^{M}_{\gamma\beta}$$

that is,

$$\Lambda_{\alpha\gamma} S^{M}_{\gamma\beta} = (q\gamma^{\mu}A_{\mu})_{\alpha\gamma} S^{M}_{\gamma\beta}.$$

The field equation and its solution

Then $\mathbf{x}$ is set to $\mathbf{x}'$ and t is set to t'.

$a\Lambda = aq\gamma^{\mu}A_{\mu}$ since $S^{M}_{\rho\sigma}(0,0) = \delta_{\rho\sigma}a$.

$\Lambda = q\gamma^{\mu}A_{\mu}$ since $a > 0$. Therefore,

$$(i\gamma^{\mu}\partial_{\mu} - m)\Psi = 0. \qquad (9)$$

$$\Psi = \sum_{\mathbf{p}}\sum_{s}\frac{N}{\sqrt{V}}\sqrt{\frac{1}{1+\exp\frac{(E-\mu)}{k_B T}}}\sqrt{\frac{m}{E}}(b^{s}_{\mathbf{p}}u^{s}(p)\exp\varphi + d^{s\dagger}_{\mathbf{p}}v^{s}(p)\exp(-\varphi))$$

Where $\varphi = i(\mathbf{p}\cdot\mathbf{x} - Et)$, $E = \sqrt{m^2 + |\mathbf{p}|^2}$ and N is a factor for normalization.

The posterior quantization

- Hence the posterior quantization is

$$\{\Psi_\alpha(\mathbf{x},t),\overline{\Psi}_\beta(\mathbf{x}',t')\}=(i\gamma^\mu\partial_\mu+m)_{\alpha\beta}(-\frac{N}{(2\pi)^3}\int_{\mathbf{p}^3}\frac{e^{ip\cdot(x-x')}-e^{-ip\cdot(x-x')}}{2E}\frac{1}{(1+\exp\frac{(E-\mu)}{k_BT})}d\mathbf{p})$$

(7)

Fuzzy theory

The issues of introducing the thermal factors

- Though it seems that all the work, treating as constraint the quantization and introducing the thermal factor, have been well done, there are **some issues** such as Lorentz invariance of the Pauli-Jordan function Δ, the consistency of quantization etc.

- Especially, the limitation of the coefficients of $b_{\mathbf{p}}^{s}$, $b_{\mathbf{p}}^{s\dagger}$, $d_{\mathbf{p}}^{s}$ and $d_{\mathbf{p}}^{s\dagger}$ in the sense of (5) make it very difficult, if it not impossible, to **normalize the spinor vectors** and to **normalize the basis of plane wave solution** simultaneously.
- Therefore, some **new mathematics** must be introduced into quantum physics.

Algebra of fuzzy complex numbers

- One of most surprising discovery of quantum physics is **the wave behavior of particle** or **its probabilistic character**.
- In order to collaborate with the discovery, the **fuzzy set theory** which is a branch of well-established pure and applied mathematics is introduced.
- The fuzzy set theory has **impacted** on many braches of science and technology.

[4] H. -J. Zimmermann, *Fuzzy set theory - and its applications*, 2nd ed., Kluwer Academic Publishers, Boston, (1991).

- The notation $(f;c)$ is used to generalize the complex number c, $|f;+z\rangle$ is used to generalize $|+z\rangle$ and so on, where f **the fuzzy coefficient**, so to speak, is a nonzero complex number with ff^* less than or equal to 1.
- When the meaning of probability is **the relative frequency of events** rather than other interpretations, this restriction, $|f| \leq 1$, may be **removed**. We are in favor to remove this restriction.

For example, f may be $\alpha\sqrt{\frac{1}{(\exp\frac{(E-\mu)}{k_B T}+1)}}$

or its equivalent for **ferminon fuzzy entities** and

$\alpha\sqrt{\frac{1}{(\exp\frac{(E-\mu)}{k_B T}-1)}}$ or its equivalent for **boson fuzzy entities**. This is a convention and it will not be mentioned again.

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- Some of **the properties** of the fuzzy complex number are

$c(f;c_1)=(f;cc_1)$ and $c+(f;c_1)=(1;c+fc_1)$,

where f, c, c_1 and c_2 are complex number.

- In order to collaborate with conventional fuzzy set theory, the **sum** and the **product** of fuzzy numbers are usually defined as follow

$(f_1;c_1)+(f_2;c_2)=(f;c_1+c_2)$ and

$(f_1;c_1)\cdot(f_2;c_2)=(f;c_1c_2)$ where $f = f_1 f_2$.

- But these definitions are not useful in QFT and the distributive law is not satisfied.

- Since most complex numbers, including the real numbers, are related to probability in QFT, the fuzzy (complex) number $(f;c)$ is related to **the probability of probability**.

- Therefore, it is meaningful to take the product of the two components of a fuzzy number and hence we have the following definitions

$$(f_1;c_1)+(f_2;c_2)=(f;\frac{f_1c_1+f_2c_2}{f})$$ and

$$(f_1;c_1)\cdot(f_2;c_2)=(f;\frac{f_1f_2c_1c_2}{f}),$$

where $f = f_1$ if $|f_1|\le|f_2|$ otherwise $f = f_2$.

- Though there have been so many meaningful definitions, we prefer the following definitions because they do almost the same purpose.

$$(f_1;c_1)+(f_2;c_2)=(1;f_1c_1+f_2c_2)$$

$$(f_1;c_1)\cdot(f_2;c_2)=(1;f_1f_2c_1c_2)$$

- The definitions of the sum and the product two

fuzzy numbers make the whole system more **easily to extend** to the **infinite sum** and the **integral** of fuzzy numbers.

- It is well defined since the **integration** is

generalization of summation. Thus

$$\int (f(t);c(t))dt = (1;\int f(t)c(t)dt).$$

- Clearly, the meaningful quantity is contained in the second component of fuzzy numbers.
- It is not necessary to define any notation or operation to extract this quantity.
- We think that the whole fuzzy number is meaningful because it gives the information along with the spirit of probability or fuzziness.

- Some properties of fuzzy kets are

$$\langle z; f^* | = | f; z \rangle^* \quad \text{and}$$

$$\langle z_1; f_1 | f_2; z_2 \rangle = (f_1 f_2; \langle z_1 | z_2 \rangle).$$

- If it is necessary, then we will have the

definition $|z\rangle = |1; z\rangle$.

- Similarly, we might have the algebra of fuzzy

operators, which will not be listed here since we will not use them in QFT.

Probabilistic relation between operators and state vectors

- It should be clarified that **neither** the quantum states, such as ket and bra, **nor** the operators, such as creator and annihilator, are **probabilistic**.
- It is **that** *when, where, which particle with what momentum and what energy will be created or destructed* is **probabilistic**.

- Since that **the operator operates on a state vector** is interpreted to be a particle creation or destruction which is **a probabilistic event**, we had better to think of it as **probabilistic** or **fuzzy**, so to speak, **the relation between the operators and state vectors**.
- Though the probability of some event might be very small before it happens, **either** the probability becomes 1 after it has happened **or** it is irrelevant to talk the probability of this event after it has happened .

- Therefore, traditional notation $a|n\rangle = \sqrt{n}|n-1\rangle$ is used to describe **the event that has happened** or **assumed to be a fact** instead of a event to be predicted, and $a|n\rangle = \sqrt{n}|f_a;n-1\rangle$ is used to describe **an event to be predicted**.
- Though the outcome of $a|n\rangle$ might be **either** $\sqrt{n}|n-1\rangle$ **or** $\sqrt{n}|f_a;n-1\rangle$, there will be no ambiguities since the meaning of the context will make it clear.

The fuzzy relation and its representation

- There must be a rule which is a **correspondent** of **Rule (II)** in the previous section. This rule is called **the rule of fuzzy relation in QFT**.

- It is “ The **relation** between a non-constant polynomial or its generalization, a power series, of **the whole field operator** Ψ and $\overline{\Psi}$, but not a part of the field operator Ψ and $\overline{\Psi}$ such as $\Psi b_{\mathbf{p}}^{s\dagger}$, $\{\Psi, b_{\mathbf{p}}^{s\dagger}\}$ and $b_{\mathbf{p}}^{s\dagger}$, and the **vacuum state** $|0\rangle$ is **fuzzy relation**.
- If the polynomial of field operators Ψ and $\overline{\Psi}$ operates on $|0\rangle$, then the outcome is the combination of **fuzzy kets**, $\left|f^{k};q\right\rangle$, which are corresponding to the terms of this polynomial, where f is the thermal factor and k the number of Ψ and $\overline{\Psi}$.”

- This rule guides us how to apply the fuzzy relation to QFT and makes the application of fuzzy theory is not fuzzy.
- Therefore, **if and only if** a non-constant polynomial of **the whole field operator** Ψ and $\overline{\Psi}$, but not a part of the field operator Ψ and $\overline{\Psi}$ such as $\Psi b_{\mathbf{p}}^{s\dagger}$, $\{\Psi, b_{\mathbf{p}}^{s\dagger}\}$ and $b_{\mathbf{p}}^{s\dagger}$, **operates** on **the vacuum state**, $|0\rangle$, then **fuzzy kets** are obtained.

The thermal factor being absorbed in the fuzzy relation between operator and ket

- From the above, **the thermal factors** are **absorbed in the fuzzy relation** defined in the rule of fuzzy relation in QFT.
- Therefore, there are **no** fuzzy entities concerning if the element of the operator space **does not** operate on the element of the state vector space, Fock space.

The separation of two algebraic system

- It is assumed to be very safe that the mathematics of **operator space** and the mathematics of **state vector space** can **be formulated separately** as two independent algebraic systems.

The application of fuzzy theory to QFT

- Since the effect of **thermal factor** can **be absorbed in** the symbolization of **fuzzy relation**, the fuzzy quantities, it is **not necessary** to introduce the **thermal factor** **explicitly**.
- Now we are ready to study **the complete theory**, and **QED** is taken as **an example**.

The Lagrangian is not listed here, the **quantization** are defined by **the differential equation**.

Let

$$S^{M}_{\rho\sigma}(\mathbf{x}-\mathbf{x}',t-t')=\{\Psi_{\rho}(\mathbf{x},t),\overline{\Psi}_{\sigma}(\mathbf{x}',t')\} \quad (10)$$

$$(i\gamma^{\mu}\partial_{\mu}-m)_{\alpha\gamma}S^{M}_{\gamma\beta}(\mathbf{x}-\mathbf{x}',t-t')=0 \qquad (11)$$

with some **initial condition**

$$S^{M}_{\rho\sigma}(\mathbf{x}-\mathbf{x}',0)=\delta_{\rho\sigma}\delta(\mathbf{x}-\mathbf{x}') \qquad (12)$$

$$\{\Psi_{\rho}(\mathbf{x},t),\Psi_{\sigma}(\mathbf{x}',t')\}=\{\overline{\Psi}_{\rho}(\mathbf{x},t),\overline{\Psi}_{\sigma}(\mathbf{x}',t')\}=0$$

By applying the method of **Lagrange multiplier**, the field equation can be obtained from the Lagrangian and the quantization

$$i\gamma^{\mu}\partial_{\mu}\Psi - m\Psi + \Lambda\Psi = q\gamma^{\mu}\Psi A_{\mu} \qquad (13)$$

Multiply both sides of the equation (13) by $\overline{\Psi}(x',t')$ from the left, multiply both sides of the equation (13) by $\overline{\Psi}(x',t')$ from the right and take the sum of them. We get

$$(i\gamma^{\mu}\partial_{\mu} - m + \Lambda)_{\alpha\gamma} S^{M}_{\gamma\beta} = (q\gamma^{\mu}A_{\mu})_{\alpha\gamma} S^{M}_{\gamma\beta}. \qquad (14)$$

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From (11) and (14), we get

$$\Lambda_{\alpha\gamma} S^M_{\gamma\beta} = (q\gamma^\mu A_\mu)_{\alpha\gamma} S^M_{\gamma\beta}$$

Setting t to be t', using the result of (12) and taking the three dimensional integration of $\mathbf{x}'$, we get

$$\Lambda_{\alpha\beta} = (q\gamma^\mu A_\mu)_{\alpha\beta}.$$

The field equation and its solution

Therefore, the field equations are

$$(i\gamma^{\mu}\partial_{\mu} - m)\Psi = 0, \quad (15)$$

and

$$\Box A^{\mu} = q\overline{\Psi}\gamma^{\mu}\Psi \quad (16)$$

- The exact solutions of the similar equations were found by Y. -S. Tsai et al.[1]
- With slight modification, the exact solutions of (15) and (16) can be obtained.

- The field variables is

$$\Psi = \sum_{\mathbf{p}} \sum_{s} \frac{N}{\sqrt{V}} \sqrt{\frac{m}{E}} (b_{\mathbf{p}}^{s} u^{s}(p) \exp\theta + d_{\mathbf{p}}^{s\dagger} v^{s}(p) \exp(-\theta))$$

where $\theta = i(\mathbf{p}\cdot\mathbf{x} - Et)$, $E = \sqrt{|\mathbf{p}|^2 + m^2}$, N is a factor for normalization, T is the temperature , μ is the chemical potential and k_B is a the Boltzmann constant.

The examples

- The following are some **illustrative examples** of the application of the fuzzy theory to QFT.

- According to the conventional definitions, we **use** the solution of **free Dirac equation** and the solution of **free Maxwell equation** in the following examples.

Example (1):

Let S_3 be the **z-component** of **the spin operator**,

that is,

$$S_3 = \frac{1}{2}\int \Psi^\dagger \Sigma_3 \Psi dx.$$

Now, this operator **operates** on one electron state, we have

$$S_3 b_{\mathbf{p}}^{s\dagger}\left|0\right\rangle = \frac{m}{2E} u^{s\dagger}(p)\Sigma_3 u^s(p) b_{\mathbf{p}}^{s\dagger}\left|0\right\rangle.$$

Since it is **excluded** by **the rule of fuzzy relation in QFT**, there is **no thermal factor** at all.

Example (2):

The **electron propagators** must be

$$\langle 0|T(\Psi_\alpha(x)\overline{\Psi}_\beta(x'))|0\rangle = (1; \frac{1}{(2\pi)^4}\int d^4p \frac{1}{\exp\frac{\sqrt{|\mathbf{p}|^2+m^2}-\mu}{k_BT}+1}\frac{i(\gamma\cdot p+m)_{\alpha\beta}}{p^2-m^2+i\varepsilon}e^{-ip\cdot(x-x')})$$

Since it is a **polynomial** of field operators with **degree two**, the fuzzy relation introduce the **square** of thermal factor

$$\sqrt{\frac{1}{\exp\frac{\sqrt{|\mathbf{p}|^2+m^2}-\mu}{k_BT}+1}}.$$

Courtesy of YEONG-SHYEONG TSAI, LU-HSING TSAI, HUNG-MING TSAI and PO-YU TSAI
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Example (3):

The **photon propagator** must be

$$\langle 0|T(A_\mu(x)A_\nu(x'))|0\rangle = (1; \frac{1}{(2\pi)^4}\int d^4k \frac{1}{\exp\frac{|\mathbf{k}|}{k_B T}-1}\frac{-ig_{\mu\nu}}{(k^2+i\varepsilon)}e^{-ik\cdot(x-x')})$$

Since it is a **polynomial** of field operators with **degree two**, the fuzzy relation introduce the **square** of thermal factor $\sqrt{\dfrac{1}{\exp\frac{|\mathbf{k}|}{k_B T}-1}}$.

Courtesy of YEONG-SHYEONG TSAI, LU-HSING TSAI, HUNG-MING TSAI and PO-YU TSAI
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Discussion

Predicted phenomena

- Since the thermal factor is a function of **temperature**, most meaningful quantities derived in QFT such as **energy levels** of hydrogen atoms, **gyromagnetic ratios** of electron and muon etc., might be **dependent on** the **temperature** if the temperature is well defined.
- Hopefully, by introducing **the thermal factors**, **the ultraviolet divergence** of the quantum fields shall be **removed without** applying the renormalization theory.

The distribution before the thermal factor being introduced

- It seems very strange or aberrant to emphasize the distribution of the operators in the field variables.
- Actually, this is not true because there have already been the distribution in the creation or destruction operators of the field variables before the thermal factors are introduced.

- In the sense of the expression of Eq. (5),

$\bar{\xi}^s_p \xi^s_p$ are r_1 and $\frac{r_2}{E}$ for the Dirac field and the Maxwell field respectively, where r_1 and r_2 are constants which are independent of energy and momentum.

- Therefore, as if it were, the distribution in the Dirac field is uniform and the distribution in the Maxwell field is $\frac{r_2}{E}$.

- Unfortunately, the uniform distribution is not well defined in an unbounded region and the distribution $f(E) = \frac{1}{E}$ is not well defined either in an unbounded region.
- It is reasonable to get the conclusion that all the divergences in QFT have some connection with these undefined distributions because $\int_0^{\infty} 1 dx$, $\int_0^1 \frac{1}{x} dx$ and $\int_1^{\infty} \frac{1}{x} dx$ are divergent.

Beyond the QFT

- Though the results of this paper are derived from QED and QCD of QFT, the results **are also applicable to**, such as string theory, those theory formulated by action integral and quantization.

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