String-inspired Progress in Perturbative Gauge Theories

Radu Roiban
Pennsylvania State University

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String theory

– emerged as an attempt to gain quantitative understanding of strong interactions; states were associated to the QCD bound states

– turned out to be a unified theory of gravity and other interactions;

– QCD is still there; however, typically, fundamental excitations of strings are related to perturbative states of QCD

– through holography nonperturbative states may be accessed

String theory: long list of field theory implications/techniques

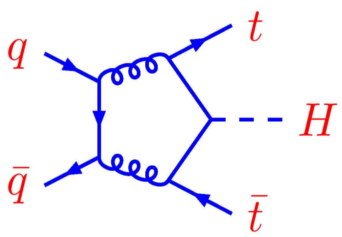
- correlation functions (gauge/string duality)
- exact effective superpotentials (topological strings)
- new perturbative techniques for gauge theories and gravity

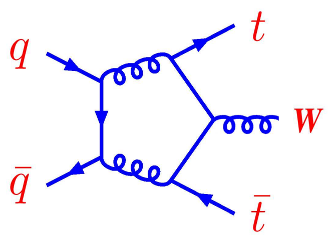
Early input:

- low energy limit of string amplitudes naturally implement gauge invariance
- KLT relations (gravity $\leftrightarrow$ gauge amplitudes)

Perturbative gauge theory calculations are essential for performing experimental tests of standard model and its extensions

State of the art analytical QCD calculation: 5-point 1-loop





Brute-force calculation

– large numbers of Feynman diagrams (6g 10^4; 7g 10^5)

– enormouns, unwieldy expressions which are numerically unstable

– integral reduction is responsible for further numerical instabilities

e.g. Gram determinants

Backgrounds to Higgs production as well as new physics at LHC require higher point amplitudes

Origin of complexity: vertices involve gauge-dependent off-shell states

Major on-shell simplifications! e.g.

$$A(1^-,2^-,3^+ ... n^+) = i\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle \ldots \langle n1\rangle} \;; \quad \langle ij\rangle = \bar{u}(k_i)\,\gamma_R\, u(k_j)$$

Parke, Taylor
Berends, Giele

$$\mathcal{A}(1^-, 2^-, 3^+ \ldots n^+) = \sum_{\text{perm's}} \text{Tr}\,[T^{a_1} \ldots T^{a_n}]\mathcal{A}(1^-, 2^-, 3^+ \ldots n^+)$$

On-shell formalisms in chronological order:

• unitarity method		Bern, Dixon, Dunbar, Kosower
• MHV vertices	trees:	Cachazo, Svrcek, Witten
	1-loop:	Brandhuber, Spence, Travaglini
• On-shell recursion	trees:	Britto, Cachazo, Feng, Witten
• recursion+unitarity		Berger, Bern, Dixon, Forde, Kosower

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Latest: Witten (2003)

The perturbative expansion of $\mathcal{N} = 4$ SYM is equivalent to the non-perturbative topological string field theory of the B-model on $\mathbb{P}^{3|4}$; a D1-instanton of genus g contributes to amplitudes with at most g loops and an instanton of degree d contributes to an amplitude with $d + L - 1$ negative helicity gluons.

- On-shell condition $p^2 = 0$ is imposed from the outset
- Natural variables – related to spinor helicity variables λ^α and $\tilde{\lambda}^{\dot\alpha}$
- Not the same however: $\bar{\lambda}^{\dot\alpha} \longrightarrow$ its Fourier-conjugate $\mu^{\dot\alpha}$

$$\tilde{\lambda}^{\dot\alpha} \qquad \longleftrightarrow \qquad \frac{\partial}{\partial \mu_{\dot\alpha}}$$

  - most natural signature $(++--)$; λ and $\tilde{\lambda}$ are independent

Main addition: use of complex momenta

- each external gluon labeled by (λ, μ) or $(\lambda, \bar{\lambda})$ and choice of helicity
  - polarization vectors can be reconstructed
- gluon wave function:

$$\tilde{A}(\lambda, \mu, \pm) = \int d\tilde{\lambda}\, e^{i[\tilde{\lambda},\mu]} A(\lambda, \tilde{\lambda}, \pm)$$

- Formal expression of string scattering amplitudes

$$\mathcal{A}_{tree} = \int d\mu_{\text{degree } d, \text{ genus } 0} \prod_{i=1}^{n} \Phi(Z(\sigma_i))\, d\sigma_i \, \langle J(\sigma_1) \ldots J(\sigma_n) \rangle_{g=0}$$

Connected instantons: (Spradlin, Volovich, RR)

– several different forms, useful for different purposes

- Tree-level color-ordered S-matrix:

$$A_n = \sum_{d=1}^{n-3} \int d\mathcal{M}_{n,d}\, J \prod_{i=1}^{n} \delta^2(\lambda_i^\alpha - \xi_i P_i^\alpha) \prod_{k=0}^{d} \delta^2(\sum_{i=1}^{n} \xi_i \sigma_i^k \tilde{\lambda}_i^{\dot\alpha}) \delta^4(\sum_{i=1}^{n} \xi_i \sigma_i^k \eta_{iA})$$

integral over
moduli space of curves
with some measure J

$P_i^\alpha = \sum_{k=0}^{d} a_k^\alpha \sigma_i^k$
degree d polynomial

$$A_n, = \sum_{d=1}^{n-3} \sum_{q|H_r^A=0} \delta^4(H_1^A)\, J \frac{(\det F)^4}{\det(\partial \widehat{H}_r^A/\partial q_s)} \qquad J = \frac{1}{\prod \xi_i} \frac{J_0}{\prod(\sigma_i - \sigma_{i+1})}$$

- H_r^A= arguments of first two sets of delta functions
- $q_s = (\sigma_i, \xi_i, a_k^\alpha)$ $\# = (n + n + 2(d+1)) - 4$
- $F_i^{\ k} = \xi_i \sigma_i^k$; i runs over negative hel. gluons

This formula is appealing for several reasons

- as stringy as it gets
  - more of combinatorial type
  - suggests there should be a precise topological string interpretation
- uniformly covers all helicity orderings
  - the same solution determines all amplitudes with fixed n and n_-
- easily modified to accommodate scalar and fermion fields
  - if i, j are fermions then $(\det F\big|_{\hat{j}})^4 \to (\det F)^3 \det F\big|_{i\to j}$
- potential use in conjunction with unitarity method
  - Only numerator factor is sensitive to type of particles and helicity
  - numerator factor: $(\det F_1\big|_{\hat{i}} \det F_2\big|_{\hat{j}} - \det F_1\big|_{\hat{j}} \det F_2\big|_{\hat{i}})^4$

Disonnected instantons: (Cachazo, Svrcek, Witten)

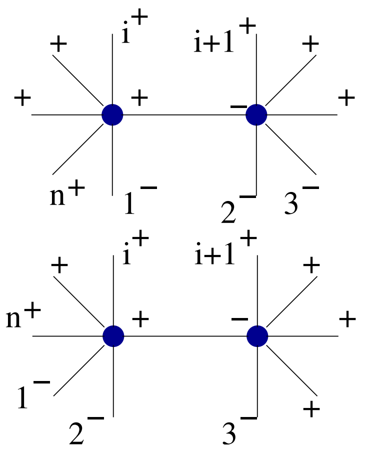
$$A_n = \sum_{n_-} \sum_{\mathcal{D}|\text{card}\mathcal{D}=n_- -1} A_k^{\text{MHV}} \prod_{\{ij\}\in\text{Links}} \frac{1}{P_{ij}^2}$$

- Reproduces gauge theory amplitudes

◇ relation w/ connected instantons: localization (Gukov, Neitzke, Motl)

MHV diagrams (Cachazo, Svrcek, Witten)

Gauge theory amplitudes can be computed by sewing together with Feynman propagators off-shell-continued MHV amplitudes: $\langle kP\rangle = \epsilon_{ab}\lambda_k^a P^{a\dot{b}}\tilde{\eta}_{\dot{b}}$ – arbitrary η



$$\sum_{i=3}^{n-1} \left[\frac{\langle 1P_i\rangle^3}{\langle P_i, i+1\rangle\langle i+1, i+2\rangle \ldots \langle n1\rangle}\right] \frac{1}{P_i^2} \left[\frac{\langle 23\rangle^3}{\langle P_i 2\rangle \ldots \langle iP_i\rangle}\right]$$

$$+ \sum_{i=3}^{n-1} \left[\frac{\langle 12\rangle^3}{\langle 2P_i\rangle\langle P_i, i+1\rangle \ldots \langle n1\rangle}\right] \frac{1}{P_i^2} \left[\frac{\langle 34\rangle^3}{\langle P_i 2\rangle \ldots \langle iP_i\rangle}\right]$$

- the arbitary spinor η used to define the off-shell continuation drops out of the final result

  $\longrightarrow$ Lorentz invariance is restored

- the unphysical poles $1/\langle iP \rangle$ are spurious

- Correct multi-particle singularities

◇ Amplitudes computed from MHV vertices have the same poles and residues as the amplitudes computed from regular Feynman diagrams $\longrightarrow$ at tree level this guarantees that the two are the same

◇ Vastly more efficient: e.g. 6g 220 Feynman diag. vs. 6 MHV diag.

◇ Attempts to relate MHV rules to (S)YM Lagrangian

(Gorsky; Masfield; Boels, Mason, Skinner)

- Still... for given N and n_-, number of MHV diagrams grows as

$$N = \frac{1}{n_-}\binom{n-3}{n_--2}\binom{n+n_--3}{n_--2} \sim n^{2n_--4}$$

Can we do better?

Yes! Historically based on IR behavior of 1-loop amplitudes

◇ 2-term expression for 6-gluon amplitude (RR, Spradlin, Volovich) based on (Bern, DelDuca, Dixon, Kosower)

◇ 3-term expression for 7-gluon amplitude (Bern, DelDuca,Dixon, Kosower)

◇ 6-term expression for 8-gluon amplitude (RR, Spradlin, Volovich)

- Compare with 44 MHV diagrams and 34300 Feynman diagrams

$$A(1^-,2^-,3^-,4^-,5^+,6^+,7^+,8^+) = \frac{[1|k_2^{[3]}|5\rangle^3}{(p_2+p_3+p_4+p_5)^2[2\,3][3\,4][4\,5]\langle 6\,7\rangle\langle 7\,8\rangle\langle 8\,1\rangle[2|k_3^{[3]}|6\rangle} +$$

$$+ \text{ 5 similar terms}$$

- from advances in calculation of 1-loop amplitudes in $\mathcal{N}=4$ SYM and the consistency of their IR singularities

- suggested existence of quadratic on-shell recursion for tree amplit's

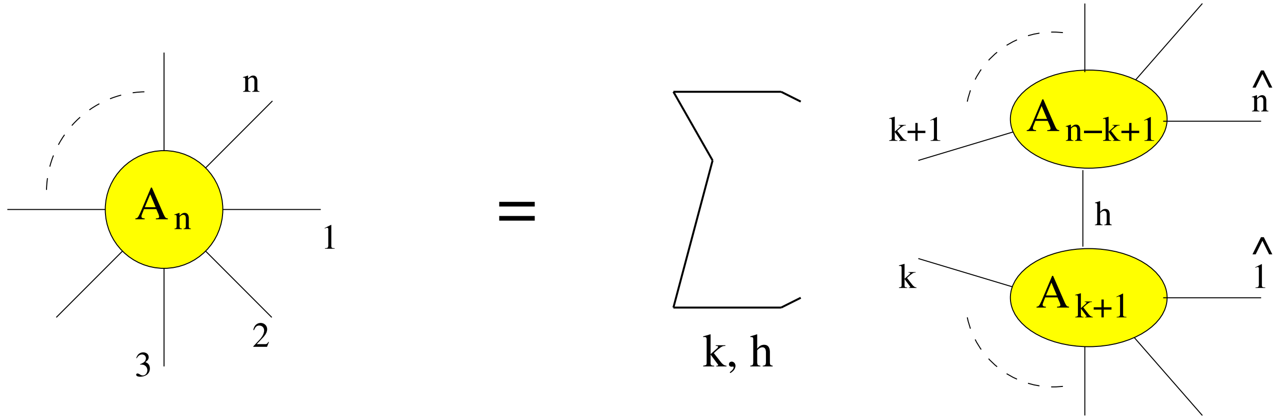
$$A_n \sim \sum_{k\geq 2} A_{k+1} A_{n-k+1}$$

- The recursion:

  derived from 1-loop IR (Britto, Cachazo, Feng)

  proved using basic field theory+cpx (Britto, Cachazo, Feng, Witten)

  proved using largest time equation+spacecone (Vaman, Yao)



$$\widehat{p}_n = p_n + z\lambda_1\bar{\lambda}_n$$

$$\widehat{p}_1 = p_1 - z\lambda_1\bar{\lambda}_n$$

- Internal line for each factor given by momentum conservation

Its power comes from its generality: BCFW proof relies only on

- Cauchy's theorem
- Basic tree-level field theory factorization

◇ Applies to any QFT, e.g. massive theories (Badger,Glover,Khoze,Svrcek)

◇ Method $\rightarrow$ 1-loop rational parts (Berger, Bern, Dixon, Forde, Kosower)

- This recursion makes impossible calculations possible
  - Example: split-helicity amplitudes $A(1^-,...,q^-,(q+1)^+,...,n^+)$

(Britto, Feng, RR, Spradlin, Volovich)

$$A(1^-,\ldots,q^-,(q+1)^+,\ldots,n^+) = \sum_{S_a,S_b} \frac{N_1^3 N_2 N_3}{D_1 D_2 D_3}$$

where $S_a \subset \{2,\ldots,q-2\} S_b \subset q+1,\ldots,n-1$

$$\begin{aligned} N_1 &= \langle q|P_{q,b_1} P_{b_1+1,a_1} P_{a_1+1,b_2} \cdots P_{b_{k+1}+1,1}|1\rangle \\ D_1 &= P^2_{q,b_1} P^2_{b_1+1,a_1} P^2_{a_1+1,b_2} \cdots P^2_{b_{k+1}+1,1} \\ &\quad \text{etc.} \end{aligned}$$

- Relatively closed form for one-but-split-helicity amplitudes

Homework

Find the general alternating-helicity amplitude: $A(1^-,2^+,3^-,4^+,\ldots)$

Explicit analytic S-matrix of YM theory follows from collinear limits

MHV rules

- gluons, fermions, scalars (Georgiou, Khoze; Wu, Zhu)

- gluons, quarks (Georgiou, Glover, Khoze; Su, Wu)

- Higgs and partons (Dixon, Glover, Khoze; Badger, Glover, Khoze)

- electroweak vector boson currents (Bern, Forde, Kosower, Mastolia)

On-shell recursion

- gluons and fermions (Luo, Wen)

- massive particles (Badger, Glover, Khoze, Svrcek)

- gravity (Bedford, Brandhuber, Spence, Travaglini; Cachazo, Svrcek)

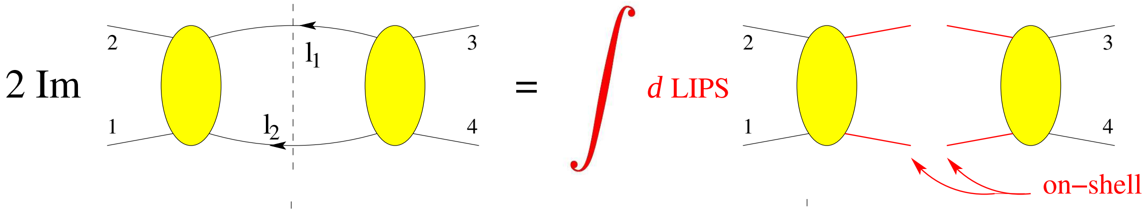
Loop level

On-shell methods existed since before twistors came along and have been used to great effect for amplitude calculations in both supersymmetric and nonsupersymmetric theories

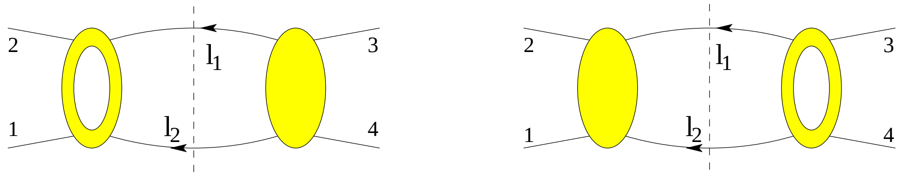
- Unitarity method (Bern, Dixon, Dunbar, Kosower)

Idea: reconstruct the amplitude from its cuts

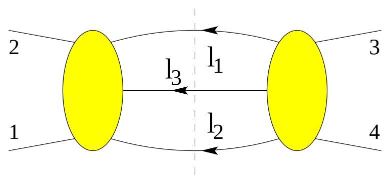
1 loop : 2-particle cuts



2 loops: 2-particle cuts

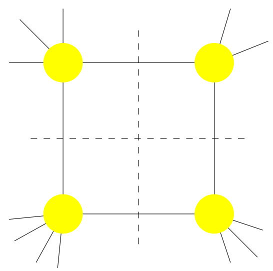


3-particle cuts



- Generalized unitarity: cut more than 2 propagators

  cut propagators = not canceling (Bern, Dixon, Kosower)



Recent progress:

Maximal supersymmetry:

- MHV vertices can be sewn into loops (Brandhuber, Spence, Travaglini)

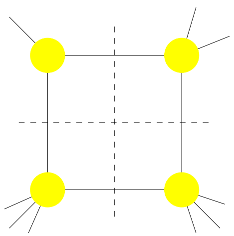
  analogous to tree-level disconnected instantons

  have this interpretation from standpoint of twistor string
  (Bena, Bern, Kosower, RR)

- generalized unitarity + basis of functions if known

  - $\mathcal{N} = 4$: all amplitudes are sums of box-integrals; need coefficients

    - quadruple cuts freeze loop integral $\longrightarrow$ read coefficient (BCF)
    - use independent λ and $\tilde{\lambda}$: nonzero on-shell 3-point amplitudes
      - allows treatment of massless box integrals



$$\text{integral coefficient} = (A_1)(A_2)(A_3)(A_4)\Big|_{\text{solution of on-shell condition}}$$ (Britto, Cachazo, Feng)

Minimal supersymmetry:

- Coefficients of box integrals follow from generalized unitarity

- In addition to box integrals, $\mathcal{N} = 1$ amplitudes contain bubble and triangle integrals: their coefficients are also needed

- MVH rules can be sewn into loops, as for $\mathcal{N} = 4$ SYM
  (Quigley, Rozali;Bedford, Brandhuber, Spence, Travaglini)

- All $\mathcal{N} = 1$ NMHV amplitudes follow from quadruple cuts
  (Bidder, Bjerrum-Bohr, Dunbar, Perkins)

- New basis of boxes and triangles allowing computation of all coefficients from quadruple cuts (Britto, Buchbinder, Cachazo, Feng)

No supersymmetry:

Crucial difference from susy: amplitudes contain rational functions

Early approaches – Bootstrap (Bern, Dixon, Kosower)

– Use unitarity method for functions with cuts

– Use factorization properties to find rational functions

Main problem: hard to find rational w/ functions right factorization

New ingredient: analyticity in the parameter z used to prove the tree-level on-shell recursion relation

$$\begin{aligned} A(z) = & \sum \text{polylogarithms} && \leftarrow \text{unitarity method} \\ + & \sum_i \frac{\mathrm{Res_i}}{z - z_i} && \leftarrow \text{on-shell recursion} \\ + & \sum_i a_i z^i && \leftarrow \text{on-shell recursion} \end{aligned}$$

(Berger, Bern, Dixon, Forde, Kosower)

What about higher loops? Best place to start is $\mathcal{N} = 4$

- multicuts don't always localize the integral (Buchbinder, Cachazo)
- basis of integral functions not known – localize on what?

• Unitarity method works to great effect and leads to a conjecture for planar MHV amplitudes to all orders in perturbation theory

$$M_n^L = (M_n^1)^L + (\text{lower loops}) + \mathcal{O}(\epsilon) \qquad M_n^L = \frac{A_n^L}{A_n^{tree}} \qquad \text{(ABDK)}$$

- checked for $L = 2, 3$ and $n = 4$ (Anastasiou, Bern, Dixon, Kosower)
(Berk, Dixon, Smirnov)
(Cachazo, Spradlin, Volovich)

- parity-even part of $L = 2$ and $n = 5$ (Cachazo, Spradlin, Volovich)

• First potential problem appears at $L - 2$ and $n = 5$ with a parity-odd term which cannot appear for $n = 4$; the conjecture relates it to $\mathcal{O}(\epsilon)$ parity-odd term at 1-loop. Nevertheless, the conjecture...

- holds for parity-odd term (Bern, Czakon, Kosower, RR, Smirnov)

What next?

- Witten's twistor string sparked a wave of remarkable progress leading to new and efficient techniques for tree-level and 1-loop amplitude calculations

- Technology is at hand for pushing the state of the art of perturbative QCD calculations

- Loops of massive vertor fields

- Assemble cross sections

- Automation?

- Though recent developments overshadowed their twistor origins, we are still to uncover all their implications for scattering amplitudes, e.g. their twistor space structure

- Prospects appear great for continued progress both in supersymmetric and nonsupersymmetric gauge theories